Changing Tastes and Effective Consistency*

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Abstract

In a single commodity setting with changing tastes, an individual's consumption plan can be obtained using naive or sophisticated choice. We provide two sufficient conditions for when (i) the solutions are unique and agree and (ii) the common plan is representable by a non-changing tastes utility. Because the solution is not revised over time, the plan and associated preferences are referred to as being effectively consistent. Afriat-style revealed preference tests are derived. The assumption of effective consistency can mitigate the problems of vulnerability to Dutch Books, non-existence of a competitive equilibrium and the aggregation of heterogeneous agents with changing tastes. *JEL* Codes: D01, D11, D50, D90.

KEYWORDS. Naive choice, sophisticated choice, effective consistency, revealed preference, Dutch book, competitive equilibrium and aggregation.

Almost 60 years after the publication of Strotz's (1956) classic paper, there continues to be considerable interest in the question of changing tastes.[1] Following the appearance of behavioural studies showing that changing taste models can do a better job of

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[1]See, for example, Pollak (1968), Phelps and Pollak (1968), Peleg and Yaari (1973), Blackorby, *et al.* (1973, 1978) and Hammond (1976). As in each of these papers except the last, when referring to changing tastes we only consider the case of exogenous taste changes.

predicting individuals' actions,[2] a number of diverse applications and theoretical extensions have appeared.[3] The changing tastes optimisation problem can most simply be framed in a three period certainty setting with a single consumption good c_t $(t = 1, 2, 3)$ in each period t. Assume preferences in period one are defined over (c_1, c_2, c_3) triples and represented by $U^{(1)}$. Preferences in period two defined over (c_2, c_3) pairs, which can depend on c_1, are represented by $U^{(2)}$. For a fixed $c_1 = \bar{c}_1$, $U^{(1)}(\bar{c}_1, c_2, c_3)$ and $U^{(2)}(c_2, c_3|\bar{c}_1)$ differ by more than a strictly increasing transform. To determine an optimal plan, an individual can follow naive choice by using $U^{(1)}$ to make the period one consumption decision and then in period two given remaining resources, use $U^{(2)}$ to make the allocation between c_2 and c_3. Alternatively, she could follow sophisticated choice and solve the problem recursively using $U^{(2)}$ to make the allocation between c_2 and c_3 conditional on c_1 and then use $U^{(1)}$ to select c_1.

In general, there is no reason to suppose that the resulting naive and sophisticated consumption plans should agree, and so the consumer confronts the problem of which process to follow.[4] However, as Pollak (1968) observed, there is no conflict in the very special case where the changing tastes $U^{(1)}$ and $U^{(2)}$ both take the form of additively separable logarithmic utility (with arbitrary discounting). Although the consumer changes her plans with the passage of time, the naive and sophisticated plans always agree. Donaldson and Selden (1981) showed that for these preferences, the common consumption plan can be rationalised by a non-changing tastes utility $\widehat{U}$. We refer to this common plan as being **effectively consistent**, since when obtained by maximising $\widehat{U}$, rather than $U^{(1)}$ and $U^{(2)}$, the plan will not be revised over time. This result seems to contradict the general view that there exists an intertemporal utility which rationalises sophisticated choice only if preferences take the strongly recursive form $U^{(1)}(c_1, c_2, c_3) = U\left(c_1, U^{(2)}\left(c_2, c_3|c_1\right)\right)$.[5]

In this paper we show that the existence of an effectively consistent plan does not require preferences to be logarithmic, additively separable or homothetic. Two mutually exclusive sufficient conditions are given for when naive and sophisticated choice are

[2]See, for example, Ainslie (1992), Laibson (1997) and Frederick, *et al.* (2002). Mulligan (1996) provides an interesting critique.

[3]Examples of the former include Diamond and Koszegi (2003) and of the latter Luttmer and Mariotti (2006, 2007).

[4]Phelps and Pollak (1968) and Peleg and Yaari (1973) argue that one should think of the problem as being equivalent to a game between two divergent individuals, myself today and myself tomorrow. Harris and Laibson (2001) assume sophisticated quasi-hyperbolic consumers. Caplin and Leahy (2006) argue that the sophisticated approach is preferable to the game theoretic models.

[5]See Remark 2 below.

unique and agree and the common plan can be rationalised by a non-changing tastes $\widehat{U}$. Specific formulas are derived for constructing $\widehat{U}$ from the assumed changing tastes $U^{(1)}$ and $U^{(2)}$. One of the sufficient conditions assumes that $U^{(1)}$ and $U^{(2)}$ take the myopic separable form introduced in Kannai, *et al.* (2014). This form of utility implies that the consumer exhibits a strong form of two stage budgeting where the choice among commodities in a group is based on within group prices and income (the expenditure on the group becomes independent of prices of goods not in the group).[6] Concrete non-additive examples of effectively consistent preferences are provided which are of particular interest due to the widely held aversion to assuming that intertemporal utility is additively separable.[7] The second sufficient condition requires the period utilities to take a quasilinear form. For our two forms of effectively consistent preferences, we derive revealed preference tests in the spirit of Afriat (1967) and Varian (1983) such that observed demand-price pairs are consistent with maximising $\widehat{U}$.[8,9]

We demonstrate that for a popular form of the quasi-hyperbolic discounted utility model of changing tastes,[10] the optimal consumption plan is effectively consistent. Although the resulting $\widehat{U}$ is an additively separable discounted utility, the discount function is neither quasi-hyperbolic nor exponential in form. $\widehat{U}$ is shown to discount the current period more heavily than the exponential case but not as strongly as the quasi-hyperbolic utility.

While the hypothesis that an individual's future tastes can be different from those currently assumed or perceived is both intuitively plausible and in some instances, such as the case of quasi-hyperbolic discounted utility, consistent with behavioural data, it nevertheless can pose a number of challenging problems for standard economic analyses. We consider three different complications and show how the assumption of effective consistency can mitigate these challenges. First given the changing tastes $U^{(1)}$ and

[6] See Deaton and Muellbauer (1980, ch.5) for a discussion of the weaker form of two stage budgeting considered by Strotz (1957, 1959) and Gorman (1959).

[7] See Fisher (1930), Hicks (1965) and Lucas and Stokey (1984).

[8] Since, as emphasised by Kubler (2004) and discussed in Section 4 below, only spot demands and prices (and incomes) are observed over time in the form of a single "extended" observation, our tests which require more observations would need to be performed in a laboratory setting such as in Choi, *et al.* (2007).

[9] One cannot determine whether a consumer's preferences correspond to $U^{(1)}$ and $U^{(2)}$ or $\widehat{U}$ based solely on observed consumption demands, since they are the same. However if the consumption optimisation problem is reformulated as a consumption-bond optimisation where there are both one and two period bonds, then it is possible to distinguish the consumer's naive consumption and bond purchases from those based on $\widehat{U}$ (see the online Appendix I).

[10] See, for instance, Phelps and Pollak (1968) and Laibson (1997).

$U^{(2)}$, a consumer in general is vulnerable to a Dutch book or money pump sequence of trades. However if the consumer's preferences are effectively consistent, then in a market setting she cannot be manipulated to impoverish herself. This result can be viewed as an alternative to the requirement in Laibson and Yariv (2007) that both $U^{(1)}$ and $U^{(2)}$ must be time separable. A second complication is that in the absence of transitive intertemporal preferences, Gabrieli and Ghosal (2013) have shown that a representative agent competitive equilibrium can fail to exist. Luttmer and Mariotti (2006, 2007) avoid this problem by assuming the intertemporal utilities $U^{(1)}$ and $U^{(2)}$ are both additively separable. Alternatively for the case where a representative agent $\widehat{U}$ exists, one can accommodate changing tastes without having to confront the possibility of the nonexistence of a competitive equilibrium.[11] Moreover given the existence of a $\widehat{U}$, one can often significantly simplify the characterisation of the equilibrium by using the first order conditions based on $\widehat{U}$. Third in economies where tastes do not change, well-known conditions exist such that the aggregate demands for a collection of agents can be rationalised by a well-behaved utility function or aggregator.[12] It is natural to ask whether the aggregate naive or sophisticated demands of consumers exhibiting changing tastes can be rationalised by an aggregator. We provide sufficient conditions such that effectively consistent preferences of the individual agents can be aggregated for both the myopically separable and quasilinear cases. Moreover, we provide explicit formulas for constructing the aggregator from the changing tastes $U^{(1)}$ and $U^{(2)}$ of the individual agents.

The rest of the paper is organised as follows. In the next section, notation and some preliminary definitions are given. Section 2 provides a motivating example. In Section 3, we derive two sufficient conditions for effectively consistent plans. Section 4 gives a revealed preference test for the myopic separable utility associated with effectively consistent preferences. Section 5 considers quasi-hyperbolic discounted utilities. In Section 6, we consider (i) the existence of Dutch Books or money pumps, (ii) naive and sophisticated equilibria and (iii) aggregation where consumers exhibit changing tastes. The last section contains concluding comments. Selected proofs are provided in the Appendix of this paper and the remaining proofs and supplemental materials are available in an online Appendix.

[11] Herings and Rohde (2006) propose specific modifications of the classic general equilibrium and Pareto Optimality notions to accommodate changing tastes.

[12] See the classic papers of Gorman (1953) and Chipman (1974) as well as the discussion of more contemporary work in Chipman (2006) and Chiappori and Ekeland (2011).

1. Preliminaries: Changing Tastes

Assume a single consumption good, certainty setting in which a consumer is endowed with income or initial wealth of y_1 which she seeks to allocate over time periods $t = 1, 2, 3$.[13] Let c_t and p_t denote, respectively, consumption in period t and the present value price in period one of consumption in period t. Preferences for periods one and two are represented respectively by

$$U^{(1)}(c_1, c_2, c_3) : C_1 \times C_2 \times C_3 \to \mathbb{R}$$

and[14]

$$U^{(2)}(\, c_2, c_3 |\ c_1) : C_2 \times C_3 \to \mathbb{R}, \quad \forall c_1 \in C_1,$$

where C_t denotes the set of possible consumption values in period t, which is (a subset of) $\mathbb{R}_+$. Both $U^{(1)}$ and $U^{(2)}$ are assumed to satisfy the following property throughout this paper.

Property 1. *The utility U is (i) a real-valued function defined on (a subset of) the positive orthant of a Euclidean space, (ii) C^2, (iii) strictly increasing in each of its arguments and (iv) strictly quasiconcave.*

At the heart of time inconsistency is the notion of changing tastes.

Definition 1. *A consumer's tastes will be said to have changed if and only there exists a $\bar{c}_1 \in C_1$ such that for every strictly increasing transformation T*

$$U^{(2)}(\, c_2, c_3 |\ \bar{c}_1) \neq T(U^{(1)}(\bar{c}_1, c_2, c_3)).$$

It is clear from this definition that whether or not preferences change is the absence or presence of a very special nesting of $U^{(2)}$ in $U^{(1)}$.

Proposition 1. *(Blackorby, et al., 1973) Given preferences corresponding to $U^{(1)}(c_1, c_2, c_3)$ and $U^{(2)}(\, c_2, c_3 |\ c_1)$, the necessary and sufficient condition for tastes not to change in the sense of Definition 1 is that for any given $\bar{c}_1 \in C_1$, there exists a strictly increasing transformation T such that*

$$U^{(1)}(\bar{c}_1, c_2, c_3) = T\left(U^{(2)}(\, c_2, c_3 |\ \bar{c}_1)\right). \tag{1}$$

[13]The assumption of three periods is made for simplicity. The general T period case is discussed in the online Appendix G.

[14]Since $U^{(2)}$ can depend on c_1 as a fixed parameter, we use $U^{(2)}(\, c_2, c_3 |\ c_1)$ for the general case. For situations where $U^{(2)}$ is independent of c_1, $U^{(2)}\,(c_2, c_3)$ is used.

To define consistent choice or planning, suppose the consumer faces the following two optimisation problems:[15]

$$P_1: \quad \max_{c_1,c_2,c_3} U^{(1)}(c_1, c_2, c_3) \qquad S.T.\ y_1 \geq p_1c_1 + p_2c_2 + p_3c_3 \tag{2}$$

and

$$P_2: \quad \max_{c_2,c_3} U^{(2)}(c_2, c_3|\ c_1) \qquad S.T.\ \ y_1 - p_1c_1 \geq p_2c_2 + p_3c_3. \tag{3}$$

Let $\mathbf{c}^\circ = (c_1^\circ, c_2^\circ, c_3^\circ)$ denote the optimal three period consumption plan for P_1. Applying terminology from Machina (1989) and McClennen (1990), the $\mathbf{c}^\circ$ plan is said to be resolute if and only if the consumer does not modify her (c_2°, c_3°) plan even if her tastes change.

The naive and sophisticated choice models for solving these two problems, where no assumption is being made about whether or not preferences change, are defined as follows.

Definition 2. *P_1 and P_2 are said to be solved by naive choice if P_1 is solved for optimal $c_1^* = c_1^\circ$ and then optimal c_2^* and c_3^* are solved via P_2 conditional on c_1^*.*

Definition 3. *P_1 and P_2 are said to be solved by sophisticated choice if P_2 is solved for conditionally optimal $c_2^{**}(c_1)$ and $c_3^{**}(c_1)$ and then optimal c_1^{**} is determined from solving P_1 conditional on $c_2^{**}(c_1)$ and $c_3^{**}(c_1)$.*

The vectors $\mathbf{c}^* = (c_1^*, c_2^*, c_3^*)$ and $\mathbf{c}^{**} = (c_1^{**}, c_2^{**}, c_3^{**})$ denote respectively the solutions resulting from the naive and sophisticated choice procedures.[16] Given the assumptions on $U^{(1)}$ and $U^{(2)}$, whereas $\mathbf{c}^*$ will be unique $\mathbf{c}^{**}$ need not be (see Blackorby, *et al.*, 1973, p. 245). A time consistent plan is defined as follows.

Definition 4. *A consumption plan (c_2°, c_3°) which optimises P_1 is said to be consistent if and only if $(c_2^\circ, c_3^\circ) = (c_2^*, c_3^*)$ for any (p_1, p_2, p_3, y_1).*

Together Definitions 1 and 4 imply that in a certainty setting, a consumption plan will be consistent if and only if the $U^{(1)}$ and $U^{(2)}$ used to solve P_1 and P_2 are equivalent up to an increasing transform.

[15] Although here the investment element of a consumption plan is ignored, in Subsection 6.2 we modify the budget constraints to allow for the investment in one and two period bonds.

[16] As pointed out by Peleg and Yaari (1973), the sophisticated choice process need not always generate an optimal plan. This problem arises when substitution of the P_2 solution into the P_1 optimisation results in $U^{(1)}$ not being concave in c_1. Consistent with Peleg and Yaari (1973, fn.1), it follows from Blackorby, *et al.* (1978) that a sufficient condition for a sophisticated solution to exist is that $U^{(2)}$ is homothetic.

Proposition 2. *(Blackorby, et al., 1973) Assume a consumer confronts choice problems P_1 and P_2. Then her consumption plan will be consistent if and only if $U^{(1)}$ and $U^{(2)}$ satisfy eqn. (1).*

In standard intertemporal choice problems only $U^{(1)}$ is specified and since the continuation of $U^{(1)}$ can be viewed as the period two utility $U^{(2)}$, tastes do not change.

2. Definitions and Motivating Example

Based on the definitions in the prior subsection, if the consumption plan is consistent, then $\mathbf{c}^* = \mathbf{c}^{**}$.[17] However, it follows from the changing tastes examples in Pollak (1968) and Donaldson and Selden (1981), where $U^{(1)}$ and $U^{(2)}$ are log additive but with different discount functions, that the consistency of a consumption plan is sufficient but not necessary for (i) $\mathbf{c}^* = \mathbf{c}^{**}$ and (ii) the existence of a non-changing tastes utility which rationalises the common plan.[18] Consider the following definition.

Definition 5. *Given $\left(U^{(1)}, U^{(2)}\right)$, if there exists a unique naive and sophisticated pair $(\mathbf{c}^*, \mathbf{c}^{**})$ as characterised in Definitions 2 and 3 which for every (p_1, p_2, p_3, y_1) satisfies $\mathbf{c}^* = \mathbf{c}^{**}$ and is rationalisable by a non-changing tastes $\widehat{U}$ satisfying Property 1, i.e.,*

$$\mathbf{c}^* = \mathbf{c}^{**} = \underset{c_1,c_2,c_3}{\arg\max}\ \widehat{U}(c_1, c_2, c_3) \quad S.T. \quad y_1 \geq p_1c_1 + p_2c_2 + p_3c_3,$$

then this common plan is said to be ***effectively consistent****. Otherwise, the plan is effectively inconsistent.* [19,20]

The reason for referring to $\mathbf{c}^*$ and $\mathbf{c}^{**}$ as being effectively consistent is that if a non-changing tastes $\widehat{U}$ exists, the agent will never revise her period one plan based on $\widehat{U}$ in

[17] It should be noted that if the consumption plan is consistent, then since $U^{(1)}$ is strictly quasiconcave, it follows from Blackorby, *et al.* (1973, Theorem 6) that $\mathbf{c}^{**}$ is unique and $\mathbf{c}^{**} = \mathbf{c}^*$.

[18] Although Pollak (1968) realised the consistency of a consumption plan is not necessary for $\mathbf{c}^* = \mathbf{c}^{**}$, he never discussed the existence of a non-changing tastes utility which rationalises the common plan.

[19] Given our assumption that $U^{(1)}$ and $U^{(2)}$ are strictly quasiconcave, the naive plan always exists and is unique. The uniqueness of the sophisticated plan follows from the definition of effective consistency. If the sophisticated plan is not unique, then the plan is said to be effectively inconsistent. This is analogous to the case of consistent plans where non-uniqueness of sophisticated plans is associated with inconsistency (see Blackorby, *et al.*, 1973, Theorem 6).

[20] Our notion of effectively consistent plans is very different from Hammond's (1976) concept of essentially consistent preferences. Hammond in effect argues that consistency and essential consistency are almost equivalent when he states 'It seems that an essential inconsistency is almost certain to occur unless ... the dynamic utility function . . . [is] ... fully consistent'. (Hammond 1976, p. 171)

period two. If the plan is consistent, $\mathbf{c}^*$ and $\mathbf{c}^{**}$ can be rationalised by $U^{(1)}$ and hence the plan is also effectively consistent. If the plan is inconsistent, one may still be able to rationalise $\mathbf{c}^*$ and $\mathbf{c}^{**}$ implying that the plan is effectively consistent as illustrated below.

Example 1. *Assume the following period one and two utilities*

$$U^{(1)}(c_1, c_2, c_3) = (c_1 c_2)^{\frac{1}{4}} + \sqrt{c_1 c_3},$$

$$U^{(2)}(c_2, c_3) = -\frac{c_2^{-\delta}}{\delta} - \frac{c_3^{-\delta}}{\delta}, \qquad \delta > -1, \quad \delta \neq 0,$$

where both satisfy Property 1. Considering naive choice first, it follows from the first order conditions for problem P_1 that

$$c_1^\circ = c_1^* = \frac{y_1}{2p_1} \qquad \textit{and} \qquad \frac{(c_2^\circ)^{-\frac{3}{4}}}{(c_3^\circ)^{-\frac{1}{2}}} = 2\left(\frac{y_1}{2p_1}\right)^{\frac{1}{4}} \frac{p_2}{p_3},$$

implying that c_2° and c_3° are nonlinear in income. Solving problem P_2, one obtains

$$c_2^* = \frac{y_1}{2\left[p_2 + p_3\left(\frac{p_2}{p_3}\right)^{\frac{1}{1+\delta}}\right]} \qquad \textit{and} \qquad c_3^* = \frac{y_1\left(\frac{p_2}{p_3}\right)^{\frac{1}{1+\delta}}}{2\left[p_2 + p_3\left(\frac{p_2}{p_3}\right)^{\frac{1}{1+\delta}}\right]},$$

which are linear in income. Therefore, we have $(c_2^\circ, c_3^\circ) \neq (c_2^*, c_3^*)$ and the consumption plan is inconsistent. Next to apply the sophisticated choice strategy, solve P_2 resulting in the period two conditional demands

$$c_2(c_1) = \frac{y_1 - p_1 c_1}{p_2 + p_3\left(\frac{p_2}{p_3}\right)^{\frac{1}{1+\delta}}} \qquad \textit{and} \qquad c_3(c_1) = \frac{\left(\frac{p_2}{p_3}\right)^{\frac{1}{1+\delta}}(y_1 - p_1 c_1)}{p_2 + p_3\left(\frac{p_2}{p_3}\right)^{\frac{1}{1+\delta}}}.$$

Maximising

$$U^{(1)}(c_1, c_2(c_1), c_3(c_1)) = \left[\frac{(y_1 - p_1 c_1)\, c_1}{p_2 + p_3\left(\frac{p_2}{p_3}\right)^{\frac{1}{1+\delta}}}\right]^{\frac{1}{4}} + \sqrt{\frac{\left(\frac{p_2}{p_3}\right)^{\frac{1}{1+\delta}}(y_1 - p_1 c_1)\, c_1}{p_2 + p_3\left(\frac{p_2}{p_3}\right)^{\frac{1}{1+\delta}}}}$$

with respect to c_1 yields

$$c_1^{**} = \frac{y_1}{2p_1}, \qquad c_2^{**} = \frac{y_1}{2\left[p_2 + p_3\left(\frac{p_2}{p_3}\right)^{\frac{1}{1+\delta}}\right]} \qquad \textit{and} \qquad c_3^{**} = \frac{y_1\left(\frac{p_2}{p_3}\right)^{\frac{1}{1+\delta}}}{2\left[p_2 + p_3\left(\frac{p_2}{p_3}\right)^{\frac{1}{1+\delta}}\right]}. \qquad (4)$$

Thus even though the consumption plan is inconsistent, because $\mathbf{c}^* = \mathbf{c}^{**}$ *it is effectively consistent. Moreover, it is straightforward to verify that the common naive and sophisticated demand functions can be rationalised by the following non-changing tastes three period utility function*

$$\widehat{U}\left(c_1, c_2, c_3\right) = c_1\left(c_2^{-\delta} + c_3^{-\delta}\right)^{-\frac{1}{\delta}}, \tag{5}$$

which satisfies Property 1.

3. Effective Consistency: Sufficient Conditions

Building on Example 1, we next derive a general sufficient condition for the consumer's plan to be effectively consistent. But first note that in the example, optimal c_1^* is independent of (p_2, p_3) which corresponds to the standard notion of a myopic plan (Kurz 1987, p. 579).[21]

Definition 6. *Given the pair* $\left(U^{(1)}, U^{(2)}\right)$ *and the budget constraint*

$$y_1 \geq p_1 c_1 + p_2 c_2 + p_3 c_3,$$

optimal period one consumption, c_1°, c_1^* *or* c_1^{**}, *is said to be myopic if and only if it is independent of* p_2 *and* p_3.

The following provides the necessary and sufficient restriction on $U^{(1)}$ such that period one consumption is myopic in the sense of Definition 6.

Proposition 3. *(Kannai, et al., 2014) Assume the optimisation problems are defined by* P_1 *and* P_2. *Then* c_1° (c_1^*) *is myopic if and only if* $U^{(1)}$ *takes the following myopic separable form*

$$U^{(1)}(c_1, c_2, c_3) = f\left(g\left(c_1\right) c_2, g\left(c_1\right) c_3\right), \tag{6}$$

where f *and* g *satisfy Property 1 and* $g > 0$.

The utility (6) is referred to as being myopic separable since it is separable in c_1 and results in myopic c_1-demand. It will be noted that in Example 1, c_1° and c_1^* are myopic.

Using this result, the following provides a sufficient condition for when a consumption plan will be effectively consistent.

[21] Although Strotz (1956) and Hammond (1976), among others, use the terms myopic and naive planning interchangeably, we distinguish these notions using Definition 6.

Proposition 4. *Given the optimisation problems P_1 and P_2, assume*

$$U^{(1)}(c_1, c_2, c_3) = f^{(1)}(g(c_1)\, c_2, g(c_1)\, c_3) \tag{7}$$

and

$$U^{(2)}(c_2, c_3 |\ c_1) = f^{(2)}(g(c_1)\, c_2, g(c_1)\, c_3), \tag{8}$$

where $f^{(i)}$ $(i = 1, 2)$ and g satisfy Property 1 and $g > 0$. Then there exists a unique $\mathbf{c}^{**}$ satisfying, for any (p_1, p_2, p_3, y_1), $\mathbf{c}^{**} = \mathbf{c}^{*}$, where the common plan is effectively consistent and can be rationalised by

$$\widehat{U}(c_1, c_2, c_3) = f^{(2)}(g(c_1)\, c_2, g(c_1)\, c_3), \tag{9}$$

which satisfies Property 1.[22]

Several observations should be made relating to the utility functions in Proposition 4. First, it will be noted that in Example 1, $f^{(1)}(x, y) = x^{\frac{1}{4}} + \sqrt{y}$, $f^{(2)}(x, y) = \left(x^{-\delta} + y^{-\delta}\right)^{-\frac{1}{\delta}}$ and $g(c_1) = c_1$. Second, the period two utility (8) depends on period one consumption c_1 implying that in general the marginal rate of substitution between period two and three consumption depends on the consumption history. Third if $U^{(1)}$ and $U^{(2)}$ correspond to different preferences and exhibit changing tastes, the functions $f^{(1)}$ and $f^{(2)}$ must differ by more than a transform, $f^{(2)} \neq T \circ f^{(1)}$ where $T' > 0$. Fourth, it follows from the proof of the proposition that when solving P_1, the c_1-value which maximises the period one utility (7) satisfies

$$p_1 c_1 + \frac{p_1 g(c_1)}{g'(c_1)} = y_1,$$

which is independent of the form of $f^{(1)}$ and $U^{(2)}$ and of (p_2, p_3). Thus consistent with Proposition 3, the consumer can simplify her period one consumption decision by ignoring (p_2, p_3) and $U^{(2)}$ and just optimising $U^{(1)}$ with regard to c_1. The resulting c_1-value corresponds to both optimal naive and sophisticated choice and is optimal with regard to $\widehat{U}$.

Although in general $U^{(2)}$ in Proposition 4 depends on c_1, we next consider whether it is possible for $U^{(2)}$ to be independent of c_1. This can be achieved when $f^{(2)}$ is homogeneous. Assuming $f^{(2)}$ is homogeneous of degree α, we have

$$U^{(2)}(c_2, c_3 |\ c_1) = f^{(2)}(g(c_1)\, c_2, g(c_1)\, c_3) = [g(c_1)]^{\alpha} f^{(2)}(c_2, c_3),$$

which is affinely equivalent to $f^{(2)}(c_2, c_3)$. Before formally stating this result as a corollary, it will prove convenient to introduce the following definition.

[22] Unless indicated otherwise, proofs are provided in the Appendix to this article.

DEFINITION 7. *For any homothetic utility function U, L_U is the strictly increasing transformation of U which results in $L_U \circ U$ being homogeneous of degree* 1.

COROLLARY 1. *Given the optimisation problems P_1 and P_2, assume*

$$U^{(1)}(c_1, c_2, c_3) = f(g(c_1)c_2, g(c_1)c_3),$$

where f and g satisfy Property 1 and $g > 0$. If $U^{(2)}(c_2, c_3)$ is homothetic, then there exists a unique $\mathbf{c}^{**}$ satisfying, for any (p_1, p_2, p_3, y_1), $\mathbf{c}^{**} = \mathbf{c}^*$. The common plan is effectively consistent and can be rationalised by

$$\widehat{U}(c_1, c_2, c_3) = g(c_1)\, u^{(2)}(c_2, c_3), \tag{10}$$

where $u^{(2)} = L_{U^{(2)}} \circ U^{(2)}$ and $\widehat{U}$ satisfies Property 1.

REMARK 1. *It should be emphasised that $U^{(2)}(c_2, c_3)$ in Corollary 1 is assumed to be independent of c_1. Otherwise, when solving P_2 using sophisticated choice the assumption that $U^{(2)}$ is homothetic in c_2 and c_3 may not ensure that c_2^{**} and c_3^{**} are proportional to $y_1 - p_1 c_1^{**}$. To see this, suppose $U^{(2)}(c_2, c_3|\ c_1) = \sqrt{c_1 c_2} + c_1^{\frac{1}{4}}\sqrt{c_3}$, which is homothetic in (c_2, c_3). Computing the first order condition for P_2 and rearranging yields $c_3 = c_1^{-\frac{1}{2}}(p_2/p_3)^2 c_2$. Substituting this expression into the P_2 budget constraint yields*

$$c_2 = \frac{y_1 - p_1 c_1}{p_2 + p_3 c_1^{-\frac{1}{2}}\left(\frac{p_2}{p_3}\right)^2},$$

which is not proportional to $y_1 - p_1 c_1$ since the denominator depends on c_1. Thus below whenever $U^{(2)}(c_2, c_3)$ is homothetic, the utility will be assumed to be independent of c_1.

The Corollary 1 special case of Proposition 4 where $U^{(2)}$ is assumed to be homothetic is of particular interest given the wide spread use of homotheticity in many economic applications. We next give a necessary condition for effective consistency assuming $U^{(2)}$ is homothetic.

PROPOSITION 5. *For the optimisation problems P_1 and P_2, if $U^{(2)}(c_2, c_3)$ is independent of c_1 and is homothetic, then the sophisticated solution $\mathbf{c}^{**}$ is unique and can be rationalised by some utility function $\widehat{U}^S$ only if it satisfies for any (p_1, p_2, p_3, y_1),*[23]

$$c_3^{**}\frac{\partial c_1^{**}}{\partial p_2} = c_2^{**}\frac{\partial c_1^{**}}{\partial p_3}. \tag{11}$$

[23]Similarly, the naive solution $\mathbf{c}^*$ can be rationalised only if $c_3^*\partial c_1^*/\partial p_2 = c_2^*\partial c_1^*/\partial p_3$.

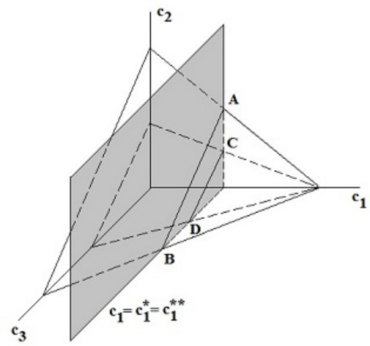

Fig. 1. *Geometry for Myopic Separable Effective Consistency*

We can consider two cases satisfying (11). The first is where $U^{(2)}$ is nested in $U^{(1)}$. In this instance since the optimal plan is consistent, the Slutsky symmetry condition is satisfied implying that (11) holds. (This can be shown using the indirect utility of $U^{(1)}$ and Roy's identity.) The second case where (11) holds is when the sophisticated period one demand is independent of period two and three prices

$$\frac{\partial c_1^{**}}{\partial p_2} = \frac{\partial c_1^{**}}{\partial p_3} = 0.$$

It follows immediately from Proposition 3 and Corollary 1 that the plan will be effectively consistent if resolute (naive) period one demand is myopic in the sense of Definition 6 and period two preferences are homothetic. Moreover in this case since $u^{(2)}$ is homogeneous of degree one, eqn. (10) can be written as

$$\widehat{U}(c_1, c_2, c_3) = g\left(c_1\right) u^{(2)}(c_2, c_3) = u^{(2)}(g(c_1)c_2, g(c_1)c_3),$$

implying that the optimal period one consumption solved from $\widehat{U}$ is also myopic.[24]

We next discuss the geometric intuition associated with c_1 being myopic using Figure 1. Assume the conditions in Corollary 1 hold.[25] Consider the two unshaded budget planes characterised by the same y_1 and p_1, but different prices p_2 and p_3. The budget lines AB and CD are drawn, respectively, on the upper and lower planes. Given that $U^{(1)}$ takes the form of (6), it follows from Proposition 3 that $c_1^{\circ} = c_1^{*}$ is myopic implying that c_1° is independent of p_2 and p_3. Thus $U^{(1)}$ determines a vertical shaded plane corresponding to $c_1 = c_1^{*}$, which intersects the two budget planes. The period two utility $U^{(2)}$ defines a set of indifference curves on the vertical plane. Tangent points

[24]It can be easily verified that when $U^{(1)}$ takes the quasilinear form in Proposition 6 below, the optimal naive period one demand is not myopic.

[25]The same discussion can also be applied to Proposition 4.

on AB and CD correspond to the naive solutions for the two budget planes.[26] On the other hand if the consumer follows sophisticated choice, then on each vertical c_1 plane there exist tangent points on the respective budget lines. Given that c_1^{**} equals c_1^*, it also is myopic, and $U^{(1)}$ determines the same shaded vertical plane corresponding to $c_1 = c_1^* = c_1^{**}$ independent of how the budget plane shifts with changing p_2 and p_3. Given that there exists a $\widehat{U}$ which rationalises sophisticated (or naive) choice, $\widehat{U}$ generates the same set of indifference curves as $U^{(2)}$ on each c_1 plane and selects the same $c_1 = c_1^* = c_1^{**}$ vertical plane as $U^{(1)}$.[27]

Given that the Proposition 4 sufficient condition for effective consistency is based on $U^{(1)}$ and $U^{(2)}$ being myopic separable, is it also the case that myopic separability is necessary? The following, based on a quasilinear $U^{(1)}$ and $U^{(2)}$, demonstrates that this is not the case.[28]

PROPOSITION 6. *Given the optimisation problems P_1 and P_2, assume that*

$$U^{(1)}(c_1, c_2, c_3) = f^{(1)}(c_1) + g^{(1)}(c_2) + c_3$$

and

$$U^{(2)}(c_2, c_3) = g^{(2)}(c_2) + c_3,$$

where $f^{(1)}$, $g^{(1)}$ and $g^{(2)}$ satisfy Property 1.[29] *Then there exists a unique $\mathbf{c}^{**}$ satisfying, for any (p_1, p_2, p_3, y_1), $\mathbf{c}^{**} = \mathbf{c}^*$, where the common plan is effectively consistent and can be rationalised by*

$$\widehat{U}(c_1, c_2, c_3) = f^{(1)}(c_1) + g^{(2)}(c_2) + c_3.$$

REMARK 2. *Blackorby, et al. (1973, Theorem 6, p. 247) state that 'an intertemporal utility function which generates the demand functions of a sophisticated society exists if and only if the society preferences are strongly recursive with a consistent representation'.*[30] *The sufficiency part is not surprising since if preferences are strongly recursive*

[26] If the plan is consistent, then $U^{(1)}$ generates the same set of indifference curves as $U^{(2)}$ on each budget plane. Otherwise, $U^{(1)}$ produces another set of indifference curves, where the different tangent points on AB and CD correspond to the resolute solution for the two budget constraints.

[27] If as in Proposition 6 below $U^{(1)}$ and $U^{(2)}$ are quasilinear in c_3, then the optimal c_1^{**} $(= c_1^*)$ does not stay on the same vertical c_1 plane when shifting the budget plane with changing p_2 and p_3 since c_1^{**} (c_1^*) is not myopic.

[28] See the online Appendix F for a discussion of normal good behaviour for changing tastes including the myopic separable and quasilinear forms of effectively consistent preferences.

[29] It should be noted that it is not necessary for $U^{(1)}$ to be quasilinear in c_3. If $U^{(2)}$ is quasilinear in c_2, then the optimal plan is effectively consistent when $U^{(1)}$ is quasilinear in c_2.

[30] A similar assertion can be also found in Blackorby, *et al.* (1978, Theorem 10.5), where they introduce additional assumptions but seem to reach essentially the same conclusion.

(i.e., satisfy eqn. (1) in Proposition 1 above) with a consistent representation, then sophisticated choice can be rationalised by a $\widehat{U} = U^{(1)}$. On the other hand, the necessity part seems to suggest that when sophisticated choice can be rationalised, preferences must be consistent. But this is contradicted by Propositions 4 and 6 and Example 1.

REMARK 3. *Although the sufficient conditions for effective consistency provided by Propositions 4 and 6 are mutually exclusive,*[31] *both require period one demand to be rationalisable by at least two different utility functions. In general this is necessary for the plan to be effectively consistent but not consistent. That is, given $U^{(1)}$ and a (resolute) optimal consumption vector $(c_1^\circ, c_2^\circ, c_3^\circ)$, one needs to find another vector (c_1°, c_2, c_3) with the same period one demand function but different period two and three demands that is generated by a $\widehat{U}$. Since there is total freedom to choose the form of the c_2 demand function (with the optimal c_3 being determined from the budget constraint), one might think that it would not be difficult to find such a demand function and utility. However for a $\widehat{U}$ to exist, the requirement that the Slutsky matrix of the new demand system be symmetric seems quite difficult to satisfy. In Propositions 4 and 6, the period one demand is either independent of both period two and three prices or independent of income and period two prices. This seems to give more freedom to satisfy the Slutsky matrix restriction. It remains an open question whether in the presence of changing tastes, the $U^{(1)}$ forms in Propositions 4 and 6 together are also necessary for the existence of an effectively consistent plan.*

Based on the results in this section, it seems natural to conjecture the following although we have not been able to prove it.

CONJECTURE 1. *Agreement of the naive and sophisticated solutions is necessary and sufficient for (i) the naive plan to be rationalised by $\widehat{U}^N$ and (ii) for the sophisticated plan to be rationalised by $\widehat{U}^S$. Moreover, the resulting $\widehat{U}^N$ and $\widehat{U}^S$ are ordinally equivalent.*[32]

[31] To see this, note that if $U^{(1)}(c_1, c_2, c_3)$ is quasilinear in c_3 as in Proposition 6, it cannot also be myopic separable as assumed in Proposition 4.

[32] Donaldson and Selden (1981) prove that if $U^{(1)}$ and $U^{(2)}$ are homothetic and the distribution of income between periods one and two is price and aggregate income independent, then the naive and sophisticated solutions can be generated by $\widehat{U}^N$ and $\widehat{U}^S$, respectively. They comment in their Remark 3 that this conclusion does not ensure that the naive solution and the sophisticated solution will give the same demand functions. To the contrary, Conjecture 1 states that one can never find a $\widehat{U}^N$ that isn't also a $\widehat{U}^S$ and vice versa.

To extend Proposition 6 to the T (> 3) period case,[33] one can simply assume that $U^{(i)}$ $(i = 1, 2, ..., T-1)$ is additively separable and quasilinear in c_T. Then the optimal plan is effectively consistent and $\widehat{U}$ is also quasilinear in c_T. The extension of Proposition 4 is more complicated and is provided in the online Appendix G. There we also generalise Propositions 4 and 6 to the case of more than one good in each period.

4. Revealed Preference Tests

In this section, we derive a revealed preference test for determining whether observed consumption and price data are consistent with the maximisation of effectively consistent preferences as characterised in Proposition 4. (A separate revealed preference test for the quasilinear effectively consistent utility in Proposition 6 and its proof are provided in the online Appendix H.) As Kubler (2004) notes, if only spot demands and prices (and incomes) are observed over time, then in principle one can only obtain a single ("extended") observation unless one uses experimental data.[34] To observe sequential choices with the same preferences but different prices and demands over a T period horizon, one can either (i) assume that the market starts over again after T periods or (ii) conduct laboratory tests where the subjects are asked to choose consumption streams in a set of scenarios characterised by different prices. In order to discuss the revealed preference test for the effectively consistent form of utility in Proposition 4 (and 6 in the online Appendix H), it will be assumed that N $(N > 1)$ different data sets are observed and the revealed preference test is performed in an laboratory setting.[35]

The utilities in both Propositions 4 and 6 should probably be referred to as "semi-parametric" rather than non-parameteric as in Varian (1983). Moreover in our test for the myopic separable form of effective consistency, we must assume $g(c_1)$ is concave in order to ensure that the utility (9) is strictly quasiconcave for any concave f function as required by the revealed preference test derived below.

Assume there are N observations of demands and prices $(\mathbf{c}^i, \mathbf{p}^i)_{i=1}^N$ with $\mathbf{c}^i \in \mathbb{R}^3_{++}$ and $\mathbf{p}^i \in \mathbb{R}^3_{++}$ for each $i = 1, ..., N$. Following the non-parametric approaches of Afriat (1967) and Varian (1983), we first review the standard definitions of the revealed preference relations, GARP (generalised axiom of revealed preference) and SARP (strong

[33] We assume the natural extension of decision problems P_1 and P_2 to T periods and assume that Property 1 is satisfied by each period's utility function.

[34] Also see Crawford and Polisson (2014).

[35] See Choi, *et al.* (2007) for an example of the use of laboratory tests to implement revealed preference tests in a static uncertainty setting.

axiom of revealed preference).

Definition 8. *An observation* $\mathbf{c}^i$ *is directly revealed preferred to a bundle* $\mathbf{c}$*, written* $\mathbf{c}^i R^0 \mathbf{c}$*, if* $\mathbf{p}^i \cdot \mathbf{c}^i \geq \mathbf{p}^i \cdot \mathbf{c}$*. An observation* $\mathbf{c}^i$ *is revealed preferred to a bundle* $\mathbf{c}$*, written* $\mathbf{c}^i R \mathbf{c}$*, if there is some sequence of bundles* $\left(\mathbf{c}^j, \mathbf{c}^k, ..., \mathbf{c}^l\right)$ *such that* $\mathbf{c}^i R^0 \mathbf{c}^j$*,* $\mathbf{c}^j R^0 \mathbf{c}^k, ..., \mathbf{c}^l R^0 \mathbf{c}$*.*

Definition 9. *The data* $(\mathbf{c}^i, \mathbf{p}^i)_{i=1}^N$ *satisfies GARP if* $\mathbf{c}^i R \mathbf{c}^j$ *implies* $\mathbf{p}^j \cdot \mathbf{c}^i \geq \mathbf{p}^j \cdot \mathbf{c}^j$*.*

Definition 10. *The data* $(\mathbf{c}^i, \mathbf{p}^i)_{i=1}^N$ *satisfies SARP if assuming* $\mathbf{c}^i R \mathbf{c}^j$ *and* $\mathbf{c}^i \neq \mathbf{c}^j$*, then* $\mathbf{c}^j R \mathbf{c}^i$ *is impossible.*

The next proposition characterises when observed demand behaviour is consistent with myopic separable effectively consistent preferences. Since the following test is not only a test of whether demand is consistent with maximising the effectively consistent utility $\widehat{U}$ in Proposition 4, but also the general myopic separable form, we state the result in terms of the general myopic separable utility u given by (12) below.

Proposition 7. *The following three conditions are equivalent:*

(i) *There exists a continuous, non-satiated utility function*

$$u\left(c_1, c_2, c_3\right) = f\left(g\left(c_1\right) c_2, g\left(c_1\right) c_3\right), \tag{12}$$

where letting $x = g\left(c_1\right) c_2$ *and* $y = g\left(c_1\right) c_3$*,* $f\left(x, y\right)$ *is strictly increasing and concave in* $\left(x, y\right)$ *and* $g\left(x_1\right)$ *is strictly increasing and concave in* x_1 *such that for all* $i = 1, ..., N$

$$\left(c_1^i, c_2^i, c_3^i\right) \in \underset{\mathbf{c} \in \mathbb{R}^3_{++}}{\arg\max}\ u\left(c_1, c_2, c_3\right) \qquad S.T. \quad \mathbf{p}^i \cdot \mathbf{c} \leq \mathbf{p}^i \cdot \mathbf{c}^i.$$

(ii) *There exist real numbers* $\left(F^i\right)_{i=1}^N$*,* $\left(G^i\right)_{i=1}^N > 0$ *and* $\left(\lambda^i\right)_{i=1}^N > 0$ *such that for all* $i, j \in \{1, 2, ..., N\}$

$$F^i \leq F^j + \lambda^j p_2^j \left(\frac{G^i}{G^j} c_2^i - c_2^j\right) + \lambda^j p_3^j \left(\frac{G^i}{G^j} c_3^i - c_3^j\right)$$

and

$$G^i \leq G^j \left[1 + \frac{p_1^j \left(c_1^i - c_1^j\right)}{p_2^j c_2^j + p_3^j c_3^j}\right].$$

(iii) *The data* $(G^i c_2^i, G^i c_3^i; p_2^i/G^i, p_3^i/G^i)$ *satisfy GARP for some choice of* G^i *that satisfies*

$$G^i \leq G^j \left[1 + \frac{p_1^j \left(c_1^i - c_1^j \right)}{p_2^j c_2^j + p_3^j c_3^j} \right].$$

In the proof of Proposition 7, we construct the following utility (see eqns. (E.6) - (E.7) in Appendix E)

$$f \left(g \left(c_1 \right) c_2, g \left(c_1 \right) c_3 \right) = \min_i \left(F^i + \lambda^i p_2^i \left[\frac{g \left(c_1 \right)}{G^i} c_2 - c_2^i \right] + \lambda^i p_3^i \left[\frac{g \left(c_1 \right)}{G^i} c_3 - c_3^i \right] \right),$$

where

$$g \left(c_1 \right) = \min_j \left(G^j \left[1 + \frac{p_1^j \left(c_1 - c_1^j \right)}{p_2^j c_2^j + p_3^j c_3^j} \right] \right).$$

Note that this utility used to rationalise the given demand data is piecewise nonlinear rather than taking the following piecewise linear form derived in the traditional Afriat's approach (see, for example, Varian 1983)

$$u \left(c_1, c_2, c_3 \right) = \min_i \left(U^i + \lambda^i p_1^i \left(c_1 - c_1^i \right) + \lambda^i p_2^i \left(c_2 - c_2^i \right) + \lambda^i p_3^i \left(c_3 - c_3^i \right) \right).$$

Since the indifference curves of the constructed utility $f \left(g \left(c_1 \right) c_2, g \left(c_1 \right) c_3 \right)$ do not have the linear segments, if the conditions in Proposition 7 are satisfied, the data $(c_1^i, c_2^i, c_3^i; p_1^i, p_2^i, p_3^i)_{i=1}^N$ must satisfy SARP which is stronger than GARP.[36]

It should be emphasised that the revealed preference test provided above can only verify whether a given set of demand data is consistent with maximising our form of effectively consistent utility. The test cannot in general distinguish between the cases where $\widehat{U}$ rationalises the consumer's demands based on $U^{(1)}$ and $U^{(2)}$ versus the case where her optimal demands are based on $\widehat{U}$ and she is consistent.[37] However if one reformulates the intertemporal consumption decision problem P_1 as an optimisation problem of allocating initial income among period one consumption and the holdings of zero coupon bonds with one and two period maturities, we show in the online Appendix I that it is possible to distinguish inconsistent naive and consistent choice.

Finally, one caveat should be noted on the use of revealed preference tests. Such analyses can only establish that observed demand data is or is not consistent with

[36] The Expected Utility test in the recent paper of Kubler, *et al.* (2014) is also a SARP test.

[37] In a quite interesting recent paper, Blow, *et al.* (2014) derive non-parametric revealed preference tests for the cases of time consistent exponential and time inconsistent quasi-hyperbolic discounted utility (for a formal characterisation of the latter, see Section 5 below). However their approach of formulating the problem as a game among intertemporal selves is different from the analysis considered in this paper.

the maximisation of a utility function with some non-parameteric or semi-parametric property. If the data is inconsistent with a particular form of utility such as the utility in Proposition 7, then one can conclude that the consumer's preferences are not represented by a utility of the hypothesised form. However, one can never tell if the utility function constructed following a revealed preference test really represents the consumer's preferences. This observation has implications for the possibility of using revealed preference tests to prove or disprove Conjecture 1 discussed above in Section 3. Suppose one finds that price and demand data corresponding to naive choice (or sophisticated choice) satisfy GARP and thus are consistent with maximisation of $\widehat{U}^N$ (or $\widehat{U}^S$). This does not imply the existence of $\widehat{U}^N$ (or $\widehat{U}^S$) and violation of our conjecture, since adding one more observation might be inconsistent with the existence of the utility.

5. Quasi-hyperbolic Discounted Utility

In this section, we discuss the implications of effective consistency for the quasi-hyperbolic discounted utility model first introduced by Phelps and Pollak (1968). Assume the period one utility function takes the following form

$$U^{(1)}\left(c_1, c_2, c_3\right) = \sum_{t=1}^{3} D\left(t\right) u\left(c_t\right),$$

where $D\left(t\right)$ is the discount function and the ratio $D\left(t\right)/D\left(t-1\right)$ is the discount factor.[38] The period two utility $U^{(2)}$ exhibits the same discounting pattern

$$U^{(2)}\left(c_2, c_3\right) = \sum_{t=2}^{3} D\left(t-1\right) u\left(c_t\right).$$

Following Strotz (1956), the plan is consistent if and only if the discount function is exponential $D\left(t\right) = \gamma^{t-1}$. However, empirical studies suggest that the decision making behaviour of individuals is not compatible with exponential discounted utility, as they tend to overweight the current time period relative to future periods.[39] This has led to the development of quasi-hyperbolic discounted utilities which in the $T = 3$ case take

[38] As is standard, the discount rate ρ_{t-1} $(t > 1)$ is defined by

$$D\left(t\right) = \frac{1}{\left(1+\rho_{t-1}\right)^{t-1}} \Leftrightarrow \rho_{t-1} = \left(\frac{1}{D\left(t\right)}\right)^{\frac{1}{t-1}} - 1.$$

[39] See, for instance, the extensive survey of Frederick, *et al.* (2002).

the following form

$$U^{(1)}(c_1, c_2, c_3) = u(c_1) + \beta \sum_{t=2}^{3} \gamma^{t-1} u(c_t) \tag{13}$$

and

$$U^{(2)}(c_2, c_3) = u(c_2) + \beta\gamma u(c_3), \tag{14}$$

where $0 < \beta, \gamma \leq 1$. (The discount function $D(t) = 1$ if $t = 1$ and $D(t) = \beta\gamma^{t-1}$ if $t > 1$ and the discount factor between periods one and two is $\beta\gamma$ and between periods two and three γ). Clearly eqn. (13) converges to the exponential discounted form when $\beta = 1$.

When $\beta \neq 1$, $U^{(2)}$ cannot be nested in $U^{(1)}$, implying that (13) and (14) exhibit changing tastes. The economic implications of the quasi-hyperbolic discounted form have been studied extensively (Laibson 1997 and Diamond and Koszegi 2003). The naive and sophisticated plans diverge and in general neither can be rationalised by a utility function. However as illustrated next, when the quasi-hyperbolic naïve and sophisticated plans converge and the common plan is effectively consistent, the common plan is optimal relative to $\widehat{U}$.

Example 2. *Assume the following period one and two quasi-hyperbolic discounted utilities*

$$U^{(1)}(c_1, c_2, c_3) = \ln c_1 + \beta\gamma \ln c_2 + \beta\gamma^2 \ln c_3 \tag{15}$$

and

$$U^{(2)}(c_2, c_3) = \ln c_2 + \beta\gamma \ln c_3, \tag{16}$$

where $0 < \beta, \gamma < 1$. Given that (15) and (16) exhibit changing tastes, it is straightforward to show that the resolute and naive consumption plans for periods two and three diverge. However since $U^{(1)}$ and $U^{(2)}$ satisfy the conditions in Proposition 4 for a plan to be effectively consistent, naive and sophisticated choice agree. As a result, the common solution can be rationalised by a discounted additive logarithmic $\widehat{U}$. To see this, note that applying Corollary 1

$$g(c_1) = c_1^{\frac{1}{\beta\gamma+\beta\gamma^2}} \quad \text{and} \quad u^{(2)}(c_2, c_3) = \left(c_2 c_3^{\beta\gamma}\right)^{\frac{1}{1+\beta\gamma}},$$

implying

$$\widehat{U}(c_1, c_2, c_3) = c_1^{\frac{1}{\beta\gamma+\beta\gamma^2}} \left(c_2 c_3^{\beta\gamma}\right)^{\frac{1}{1+\beta\gamma}},$$

which is ordinally equivalent to

$$\widehat{U}(c_1, c_2, c_3) = \ln c_1 + \frac{\beta\gamma + \beta\gamma^2}{1+\beta\gamma} \ln c_2 + \frac{(\beta\gamma + \beta\gamma^2)\beta\gamma}{1+\beta\gamma} \ln c_3. \tag{17}$$

Remark 4. *The existence of $\widehat{U}$ in this example seems to present a paradox. On the one hand the representation (17) does not exhibit changing tastes and hence its resulting demands are consistent, but on the other hand the form of the discount function is not exponential. The source of this paradox is the fact that one can either maintain the same pattern of discounting over time or maintain the same absolute discount functions in subsequent time periods.*[40] *To illustrate this distinction, note that the quasi-hyperbolic utilities $U^{(1)}$ and $U^{(2)}$, (15) and (16), preserve exactly the same discounting pattern between periods. In this case, the consumer always overweights the current period relative to future periods. Alternatively were the period two utility instead to take the form*

$$U^{(2)}(c_2, c_3) = \beta\gamma \ln c_2 + \beta\gamma^2 \ln c_3,$$

then the same period two and three discount functions would apply for $U^{(1)}$ and $U^{(2)}$. Since $U^{(2)}$ would then be nested in $U^{(1)}$, the optimal plan would be consistent. However in this case, the consumer only overweights the current period relative to future periods in period one and not in subsequent periods. When there exists a $\widehat{U}$, the discount functions carryover from period to period since one can take $U^{(1)} = \widehat{U}$

$$U^{(1)}(c_1, c_2, c_3) = \ln c_1 + \frac{\beta\gamma + \beta\gamma^2}{1+\beta\gamma} \ln c_2 + \frac{(\beta\gamma + \beta\gamma^2)\,\beta\gamma}{1+\beta\gamma} \ln c_3$$

and $U^{(2)}$ to be the continuation of $\widehat{U}$

$$U^{(2)}(c_2, c_3) = \frac{\beta\gamma + \beta\gamma^2}{1+\beta\gamma} \ln c_2 + \frac{(\beta\gamma + \beta\gamma^2)\,\beta\gamma}{1+\beta\gamma} \ln c_3,$$

implying that the consumer necessarily changes her discounting pattern with the passage of time.

It is demonstrated in the online Appendix J, that if $U^{(1)}$ and $U^{(2)}$ in Example 2 take the more general (non-log) CES (constant elasticity of substitution) quasi-hyperbolic discounted form, although $U^{(2)}$ is homothetic, $U^{(1)}$ does not satisfy the conditions in Corollary 1, $\mathbf{c}^{**} \neq \mathbf{c}^{*}$ and there exists no $\widehat{U}$.

We next compare the behaviour of discount functions over time corresponding to the quasi-hyperbolic and rationalised $\widehat{U}$ utility models. But first, we generalise the above discussion to T periods (the proof is provided in the online Appendix K).

[40] See Rasmusen (2008) for a thoughtful discussion of the general distinction between maintaining the same discount pattern or absolute discount functions.

Proposition 8. *Assume in period one the following T period quasi-hyperbolic discounted utility*

$$U^{(1)}(c_1, c_2, ..., c_T) = \ln c_1 + \beta \sum_{t=2}^{T} \gamma^{t-1} \ln c_t, \tag{18}$$

where $0 < \beta, \gamma < 1$ and in each future period utility preserves the same discounting pattern as $U^{(1)}$. Then $(c_1^*, ..., c_T^*) = (c_1^{**}, ..., c_T^{**})$ and the common solution can be rationalised by

$$\widehat{U}(c_1, c_2, ..., c_T) = \ln c_1 + \sum_{t=2}^{T} \alpha_t \ln c_t,$$

where the discount function α_t for $t = 2, ..., T$ is given by

$$\alpha_t = \frac{\beta^{t-1} \prod_{i=1}^{t-1} \sum_{j=1}^{T-i} \gamma^j}{\prod_{i=1}^{t-1} \left(1 + \beta \sum_{j=2}^{T-i} \gamma^{j-1}\right)} < \gamma^{t-1}. \tag{19}$$

For the exponential case

$$U^{(1)}(c_1, c_2, ..., c_T) = \sum_{t=1}^{T} \gamma^{t-1} \ln c_t,$$

it follows from (18) that the quasi-hyperbolic discount function for each $t = 2, ..., T$, will always be smaller than the exponential discount function γ^{t-1} so long as $\beta < 1$. Eqn. (19) implies that for each period $t \in \{2, ..., T\}$, the $\widehat{U}$ discount function α_t will also be less than the exponential discount function. But when $t \geq 2$, the relationship between α_t and $\beta\gamma^{t-1}$ is ambiguous in general.

In Figure 2, we plot the value of the discount functions or discounted value per unit of utility for the exponential, quasi-hyperbolic and $\widehat{U}$ cases where $\gamma = 0.95$ and $\beta = 0.6$. The utility $\widehat{U}$ smooths the discount functions for the quasi-hyperbolic model with the value of the $\widehat{U}$ discount functions being higher in the earlier years and smaller in the later years. Next consider limit cases. If $\beta \to 1$, the discount functions for the exponential, quasi-hyperbolic and rationalised discounted utilities all converge to the exponential curve. If $\gamma \to 1$, the value of the discount function for the exponential discounting model would be reflected in Figure 2 by a horizontal line at 1. Moreover, the value of the discount function for the quasi-hyperbolic model would always be β except for period one, implying in Figure 2 a sharp drop followed by a flat segment. In

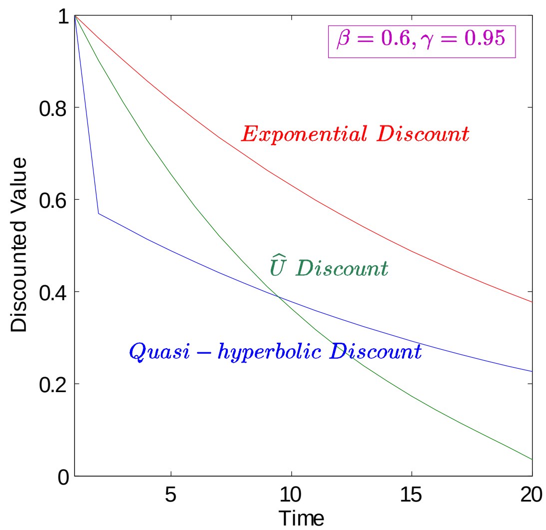

Fig. 2. *Value of Discount Functions Versus Time*

this case, the smoothed time pattern of the discount function for $\widehat{U}$ given by eqn. (19) would simplify to

$$\alpha_t = \frac{\beta^{t-1}\prod_{i=1}^{t-1}(T-i)}{\prod_{i=1}^{t-1}\left[1-\beta+\beta\,(T-i)\right]}.$$

6. Three Changing Tastes Complications

As noted earlier, changing tastes can pose a number of difficulties for standard economic analyses. In this section we consider three specific complications.

6.1. *Dutch Book Vulnerability*

An individual with intransitive preferences is said to exhibit Dutch Book or money pump behaviour if she can be induced through a sequence of trades to give up her wealth (e.g., Anand 1993). In a dynamic setting when tastes change since preferences are in general intransitive, the consumer is vulnerable to a Dutch Book. This susceptibility can occur even when the consumer's plan is effectively consistent but not consistent due to the intransitivity of intertemporal preferences. Although the consumer is vulnerable to a sequence of carefully chosen trades, Laibson and Yariv (2007) prove that in a market setting, Dutch Books cannot occur if current and future spot prices are known and fixed and preferences in each period are time separable. Assume there are H consumers, indexed by $h = 1, ..., H$. For $t = 1, 2, 3$, let $c_{h,t}$ $(\bar{c}_{h,t})$ denote consumer $h's$ consumption (endowment) in period t, p_t the price of consumption in period t and $U_h^{(1)}(c_{h,1}, c_{h,2}, c_{h,3})$ and $U_h^{(2)}(c_{h,2}, c_{h,3}|\ c_{h,1})$ the consumer's period one and two utilities.

Laibson and Yariv (2007) use the game-theoretic framework and competitive equilibrium approach of Luttmer and Mariotti (2006) and make the following price expectations assumption.[41]

Assumption 1. *Each consumer h has rational price expectations satisfying*

$$p_j^{(1)} = p_j^{(2)} = p_j \ (j = 2, 3),$$

where $p_j^{(1)}$ and $p_j^{(2)}$ denote, respectively, the price for $c_{h,j}$ in periods one and two.

Laibson and Yariv (2007) prove that if each consumer's utilities exhibit changing tastes and take the following time separable forms

$$U_h^{(1)}(c_{h,1}, c_{h,2}, c_{h,3}) = \sum_{t=1}^{3} u_{ht}^{(1)}(c_{h,t}) \quad \text{and} \quad U_h^{(2)}(c_{h,2}, c_{h,3}|\ c_{h,1}) = \sum_{t=1}^{3} u_{ht}^{(2)}(c_{h,t}),$$

where $u_{ht}^{(1)}(c_{h,t})$ and $u_{ht}^{(2)}(c_{h,t})$ are continuous, strictly increasing, and strictly concave, and each consumer's price expectations satisfy Assumption 1, then every consumer will follow an intrapersonal equilibrium strategy at each date. And there will exist a sequence of aggregate consumption demands and price vectors that correspond to an intertemporal competitive equilibrium in which Dutch Books do not exist.[42]

To show that a Dutch Book can exist in a competitive equilibrium if preferences are not time separable, Laibson and Yariv (2007, Subsection 7.1) construct an example where (adapted to our setting) a representative agent's utilities are given by

$$U^{(1)}(c_1, c_2, c_3) = c_3$$

and

$$U^{(2)}(c_2, c_3|\ c_1) = \ln(c_1 + \alpha c_2 + c_3) + c_2 \quad (0 < \alpha < 1),$$

[41] As observed by Caplin and Leahy (2006) and Gabrieli and Ghosal (2013), a sophisticated optimum, when it exists, will also be a subgame perfect equilibrium.

[42] An example in Gabrieli and Ghosal (2013) suggests that if one does not assume time separability, then the game-theoretic competitive equilibrium considered in Laibson and Yariv (2007) may not exist.

where the latter utility is not time separable. The agent faces the optimisation problems P_1 and P_2, where her budget constraint is

$$p_1\bar{c}_1 + p_2\bar{c}_2 + p_3\bar{c}_3 \geq p_1 c_1 + p_2 c_2 + p_3 c_3.$$

If one solves P_1 and P_2 following sophisticated choice, then exactly the same solution can be obtained as derived by Laibson and Yariv (2007) using the game-theoretic approach.[43] Since it is possible to find a set of endowments, a value of α and equilibrium prices such that the sophisticated optimum $(c_1^{**}, c_2^{**}, c_3^{**})$ satisfies

$$p_1\bar{c}_1 + p_2\bar{c}_2 + p_3\bar{c}_3 > p_1 c_1^{**} + p_2 c_2^{**} + p_3 c_3^{**},$$

it follows that the consumer is vulnerable to a Dutch Book.

Since time separability is only a sufficient condition for the nonexistence of a Dutch Book in a competitive equilibrium, it is natural to ask whether there are other sufficient conditions. Suppose the preferences of each consumer $h \in \{1, 2, ..., H\}$ are effectively consistent where $U_h^{(1)}$ and $U_h^{(2)}$ take the form in Propositions 4 or 6. Then consumer h's sophisticated plan can be always rationalised by a strictly quasiconcave $\widehat{U}_h$. The following proposition proves that a sophisticated equilibrium exists in which

$$p_1 c_{h,1}^{**} + p_2 c_{h,2}^{**} + p_3 c_{h,3}^{**} = p_1\bar{c}_{h,1} + p_2\bar{c}_{h,2} + p_3\bar{c}_{h,3}$$

always holds and no Dutch Book will exist (a formal proof is provided in the online Appendix L).[44]

Proposition 9. *Assume each consumer $h \in \{1, 2, ..., H\}$ in an economy solves the the optimisation problems P_1 and P_2, where her utilities $U_h^{(1)}$ and $U_h^{(2)}$ take the form in Proposition 4 or 6, and her price expectations satisfy Assumption 1. Then there exists a sophisticated competitive equilibrium in which no Dutch Book exists.*

To see the impossibility of a Dutch Book for the effective consistency case, consider the following modification of Example 1.

[43]If agents are naive, then in each period t, they will optimise $U^{(t)}$ according to current period prices. Since $U^{(t)}$ is strictly quasiconcave, we always have

$$p_1 c_{h,1}^{*} + p_2 c_{h,2}^{*} + p_3 c_{h,3}^{*} = p_1\bar{c}_{h,1} + p_2\bar{c}_{h,2} + p_3\bar{c}_{h,3},$$

implying that a Dutch Book cannot exist in a naive equilibrium. Therefore, we focus on the sophisticated equilibrium.

[44]A sophisticated competitive equilibrium is an equilibrium, where every agent follows sophisticated choice. Refer to Definition 12 below in a consumption-bond setting.

Example 3. *Assume a representative agent with the same period one and two utilities as in Example 1. Suppose that she solves the optimisation problems P_1 and P_2, where the P_1 constraint is given by*

$$p_1\overline{c}_1 + p_2\overline{c}_2 + p_3\overline{c}_3 \geq p_1 c_1 + p_2 c_2 + p_3 c_3. \tag{20}$$

The period one utility is not time separable but preferences are effectively consistent and $\widehat{U}$ takes the form given by (5). Then the sophisticated solution $(c_1^{**}, c_2^{**}, c_3^{**})$ is given by (4), where y_1 is replaced by $p_1\overline{c}_1 + p_2\overline{c}_2 + p_3\overline{c}_3$. It follows that

$$p_1 c_1^{**} + p_2 c_2^{**} + p_3 c_3^{**} = p_1\overline{c}_1 + p_2\overline{c}_2 + p_3\overline{c}_3$$

and hence there is no Dutch Book in the competitive equilibrium.[45]

6.2. *Naive and Sophisticated Equilibria*

In economies characterised by changing tastes, different notions of equilibrium have been developed to accommodate naive versus sophisticated choice behaviour (see Herings and Rohde 2006).[46] In order to examine the implications of effectively consistent preferences for naive and sophisticated exchange equilibria, we assume the former equilibrium is associated with a naive representative agent and latter with a sophisticated representative agent. To compare the equilibria, it will prove convenient to assume a consumption-bond setup where, without loss of generality, there are three time periods.[47] Let c_1, b_{12} and b_{13} denote, respectively, period one consumption, units of a one period bond that pays off one unit of consumption at the beginning of period two and units of a two period bond paying off one unit of consumption at the beginning of period three. The two period bond can be retraded in period two.[48] In both periods, consumption is the numeraire. Prices at the beginning of period one for the one

[45] Equilibrium prices can be derived from the first order conditions associated with maximising $\widehat{U}(c_1, c_2, c_3)$ subject to (20) by assuming $c_i = \overline{c}_i$ $(i = 1, 2, 3)$.

[46] Luttmer and Mariotti (2006) also discuss the existence of a competitive equilibrium where each agent has time-separable preferences and determines her optimal consumption plan using the game-theoretic approach. However since we, like Herings and Rohde (2006), consider preferences which are not necessarily time-separable, the equilibrium results in Luttmer and Mariotti (2006) are too restrictive for the case of effective consistency.

[47] Kocherlakota (2001) employs a similar setup but differs in introducing a commitment asset and assuming a game-theoretic solution.

[48] A naive agent does not anticipate changing her period one plan in period two to reflect her changing tastes and a sophisticated agent will not change her plan because she directly incorporates her changing preferences into the plan. As a result, we assume in this subsection and in the revealed preference

and two period bonds are given by, respectively, q_{12} and q_{13}. In period two, the one period bond matures. Let b_{23} and q_{23} denote, respectively, the period two units and price of the two period bond with one period of remaining maturity. Let $\bar{c}_1$, $\bar{b}_2$ and $\bar{b}_3$ denote the representative agent's endowments of period one consumption and zero coupon bonds maturing at the beginning of periods two and three, respectively.[49]

The individual's optimisation problems P_1 and P_2, (2) - (3), can be converted into the consumption-bond problems faced by the representative agent

$$Q_1 : \quad \max_{c_1, b_{12}, b_{13}} U^{(1)}(c_1, c_2, c_3) \qquad S.T.\ \bar{c}_1 + q_{12}\bar{b}_2 + q_{13}\bar{b}_3 \geq c_1 + q_{12}b_{12} + q_{13}b_{13},$$

$$S.T.\ c_2 = b_{12} \qquad \text{and} \qquad c_3 = b_{13}$$

and

$$Q_2 : \quad \max_{c_2, b_{23}} U^{(2)}(c_2, c_3) \qquad S.T.\ W_2 \geq c_2 + q_{23}b_{23},$$

$$S.T.\ c_3 = b_{23},$$

where in Q_2 period two income (or wealth) W_2 equals the period two value of bonds $b_{12} + q_{23}b_{13}$ (bought in period one, but valued in period two). This budget constraint for Q_1 is a natural extension of the constraint used in P_1. The budget constraint implied by P_2 can be written as

$$\bar{c}_1 + q_{12}\bar{b}_2 + q_{13}\bar{b}_3 - c_1 \geq q_{12}(c_2 + q_{23}c_3),$$

where the left hand side is the unconsumed wealth at the end of period one and the right hand side is the present value of future consumption. However this constraint cannot be used for Q_2, since in the presence of changing tastes the equilibrium price of the two period bond in period two q_{23} can diverge from that based on the period one implied forward rate, i.e., $q_{23} \neq q_{13}/q_{12}$. This requires us instead to use the present value of the bonds in period two based on q_{23} in the Q_2 constraint.

Given the optimisation problems Q_1 and Q_2, the naive and sophisticated representative agents will follow exactly the same solution processes described in Definitions

analysis in online Appendix I that both agents buy a portfolio of short and long term assets in period one such that there is no expectation of having to trade again in period two. This demand behaviour could, for instance, be motivated by the presence of transaction costs associated with trading in bonds. (It is clear that the presence of even the smallest transaction costs will result in both agents seeking to avoid retrading.) Without assuming this demand behaviour, the agent would be indifferent to buying different portfolios of short and long term assets given that prices do not change, resulting in the non-uniqueness of an optimal solution and a failure to have a unique equilibrium.

[49] Here we assume that $\bar{b}_2, \bar{b}_3 \neq 0$ as in Parlour, *et al.* (2011).

2 and 3 where optimal demands are based on a partial equilibrium analysis in which equilibrium prices are assumed to be given exogenously and satisfy $q_{23} = q_{13}/q_{12}$.[50] The presence of bonds does not alter the fact that if preferences are effectively consistent, the consumption plans of the naive and sophisticated agents will satisfy $\mathbf{c}^* = \mathbf{c}^{**}$. However as we next argue, the optimal bond demands b_{12} and b_{13} for the two agents will differ if the assumed $U^{(1)}$ and $U^{(2)}$ exhibit changing tastes. This is key in comparing the naive and sophisticated consumption-bond equilibria. The naive representative agent in period one follows resolute choice in determining the bond allocation

$$b_{12}^\circ = c_2^\circ \qquad \text{and} \qquad b_{13}^\circ = c_3^\circ. \tag{21}$$

In period two given that the naive agent's tastes have changed, the payoffs from the one period bond holdings b_{12}° will in general not match her desired period two optimal consumption c_2^* based on $U^{(2)}$. As a result, she will adjust her resolute two period bond holdings b_{13}° (which mature at the end of period three) to meet her optimal period two consumption requirements. The naive agent's period three consumption c_3^* will always equal her revised two period bond holdings b_{23}^*. In contrast, the sophisticated representative agent anticipates in period one that her tastes will change in period two and chooses her bond holdings to match her desired consumption in period two and three

$$b_{12}^{**} = c_2^{**} \ \text{ and } \ b_{13}^{**} = b_{23}^{**} = c_3^{**}. \tag{22}$$

The equality $b_{13}^{**} = b_{23}^{**}$ follows from the fact that the sophisticated agent does not need to retrade her two period bond holdings. Thus the period one bond allocations (b_{12}, b_{13}) will be different for the naive and sophisticated representative agents. Moreover given eqn. (22), the effectively consistent $\widehat{U}$ derived in Section 3 will not only rationalise $(c_1^{**}, c_2^{**}, c_3^{**})$ but also $(c_1^{**}, b_{12}^{**}, b_{13}^{**})$. In contrast for the naive representative agent, it follows from (21) that $(c_1^*, b_{12}^\circ, b_{13}^\circ)$ can be rationalised by $U^{(1)}$ rather than $\widehat{U}$. Thus the properties of the naive agent's effectively consistent consumption plans such as normal good behaviour cannot directly be translated to her period one bond holdings (b_{12}, b_{13}). This key difference between the agents, results in the divergence between the naive and sophisticated equilibria discussed below.

Next naive and sophisticated equilibria are defined following Herings and Rohde (2006).

[50] In the general versus partial equilibrium analysis considered below, we need to take into account the fact that the naive representative agent's assumed equality $q_{23} = q_{13}/q_{12}$ will in general fail to hold in equilibrium. In contrast, the sophisticated agent's assumption that it will hold is substantiated in the sophisticated equilibrium.

Definition 11. *An equilibrium* $(c_1, b_{12}, b_{13}, c_2, b_{23}, q_{12}, q_{13}, q_{23})$ *is a naive equilibrium if and only if the equilibrium prices are solved from* Q_1 *and* Q_2*, respectively, by setting the period one resolute demands equal to the endowments* $(\bar{c}_1, \bar{b}_2, \bar{b}_3)$ *and the period two naive demands equal to* $(\bar{b}_2, \bar{b}_3)$*.*

Definition 12. *An equilibrium* $(c_1, b_{12} = c_2, b_{13} = b_{23} = c_3, q_{12}, q_{13}, q_{23})$ *is a sophisticated equilibrium if and only if the equilibrium prices are determined by solving* Q_1 *and* Q_2 *and setting the sophisticated demands* $(c_1^{**}, c_2^{**}, c_3^{**})$ *equal to the endowments* $(\bar{c}_1, \bar{b}_2, \bar{b}_3)$*.*[51]

If the plan is effectively consistent, implying the existence of a strictly quasiconcave $\widehat{U}$ to rationalise the sophisticated representative agent's demands, then it follows from Katzner (1972) that there exists a unique sophisticated equilibrium. Given $\widehat{U}$ and the exogenous supplies $(\bar{c}_1, \bar{b}_2, \bar{b}_3)$, the sophisticated equilibrium prices can be computed directly from the first order conditions of $\widehat{U}$.[52]

The following example illustrates that assuming the naive and sophisticated representative agents' preferences are effectively consistent and their endowments are the same does not imply that the naive and sophisticated equilibria are also the same.

Example 4. *For the representative agent optimisation problems* Q_1 *and* Q_2*, assume that*

$$U^{(1)}(c_1, c_2, c_3) = \ln c_1 + \ln c_2 + \ln c_3 \quad \textit{and} \quad U^{(2)}(c_2, c_3) = -\frac{c_2^{-\delta}}{\delta} - \frac{c_3^{-\delta}}{\delta},$$

where $\delta > -1$ *and* $\delta \neq 0$*. It follows directly from Proposition 4 that* $U^{(1)}$ *and* $U^{(2)}$ *are effectively consistent and*

$$\widehat{U}(c_1, c_2, c_3) = \sqrt{c_1}\left(c_2^{-\delta} + c_3^{-\delta}\right)^{-\frac{1}{\delta}}.$$

Assume endowments are given by $\bar{c}_1$*,* $\bar{b}_2$ *and* $\bar{b}_3$*. Using the first order conditions for* $U^{(1)}$ *and* $U^{(2)}$*, naive equilibrium prices satisfy*

$$q_{13} = \frac{\bar{c}_1}{\bar{b}_3} \neq \frac{\bar{c}_1 \bar{b}_2^{-\delta}}{\bar{b}_3^{1+\delta}} = q_{12} q_{23}.$$

[51] Herings and Rohde (2006, p. 600) embed a condition analogous to $q_{12}q_{23} = q_{13}$ in their definition of a sophisticated equilibrium.

[52] The existence of a $\widehat{U}$ can be of considerable value in studying the properties of a sophisticated equilibrium. As shown in Example M1 in the online Appendix M, although one may not be able to derive an analytic expression for equilibrium prices in terms of endowments and preference parameters by equating demands and endowments, it may be relatively straightforward to do so from the first order conditions using $\widehat{U}$.

Based on the first order condition of $\widehat{U}$, sophisticated equilibrium prices satisfy

$$q_{13} = \frac{2(1+\delta)\sqrt{\overline{c}_1}}{\overline{b}_3^{1+\delta}\left(\overline{b}_2^{-\delta} + \overline{b}_3^{-\delta}\right)^{\frac{1}{\delta}+1}} = q_{12}q_{23}.$$

Thus for the sophisticated equilibrium, $q_{12}q_{23} = q_{13}$, whereas for the naive equilibrium in general $q_{12}q_{23}$ and q_{13} diverge. The intuition for why the naive and sophisticated equilibria diverge even though the representative agent's consumption plan is effectively consistent can be seen by comparing the naive and sophisticated agents' demands for the one period bond in period two. Although their consumption demands are the same, their consumption plans for periods two and three are different, which is reflected in their different one period bond demands

$$b_{12}^{*} = \frac{\overline{c}_1 + q_{12}\overline{b}_2 + q_{13}\overline{b}_3 - c_1^{*}}{2q_{12}}$$

and

$$b_{12}^{**} = \frac{\overline{c}_1 + q_{12}\overline{b}_2 + q_{13}\overline{b}_3 - c_1^{**}}{q_{12} + q_{13}\left(\frac{q_{12}}{q_{13}}\right)^{\frac{1}{1+\delta}}} = \frac{\overline{c}_1 + q_{12}\overline{b}_2 + q_{13}\overline{b}_3 - c_1^{*}}{q_{12} + q_{13}\left(\frac{q_{12}}{q_{13}}\right)^{\frac{1}{1+\delta}}}.$$

6.3. *Aggregation*

Assume there are H consumers, indexed by $h = 1, ..., H$. For $t = 1, 2, 3$, let $c_{h,t}$ $(\overline{c}_{h,t})$ denote consumer h's consumption (endowment) in period t. To simplify the notation, c_t will be used instead of $c_{A,t}$ to denote aggregate demands. Let $U_h^{(1)}(c_{h,1}, c_{h,2}, c_{h,3})$ and $U_h^{(2)}(c_{h,2}, c_{h,3} |\ c_{h,1})$ be the consumer's period one and period two utilities. The budget constraint for agent h is given by

$$p_1\overline{c}_{h,1} + p_2\overline{c}_{h,2} + p_3\overline{c}_{h,3} = y_{h,1} \geq p_1c_{h,1} + p_2c_{h,2} + p_3c_{h,3}, \tag{23}$$

where $y_{h,1}$ denotes the period one income (wealth) of consumer h. As is standard in the aggregation literature, we will make use of the following restriction on endowments.

Definition 13. *The endowments $\{\overline{\mathbf{c}}_h, ...\overline{\mathbf{c}}_H\}$ in an economy are said to be collinear if and only if they satisfy*

$$\overline{\mathbf{c}}_h = \omega_h\overline{\mathbf{c}} \qquad \forall h \in \{1, 2, ..., H\}, \tag{24}$$

where $\sum_{h=1}^{H}\omega_h = 1$ and $\overline{\mathbf{c}} = \sum_{h=1}^{H}\overline{\mathbf{c}}_h$.

Given (23), collinearity of endowments is equivalent to consumer incomes being proportional. It follows from Chipman (1974) that if each consumer has preferences which do not change over time, are homothetic and are representable by a U_h satisfying Property 1 and endowments are collinear, then there exists an aggregator U_A which satisfies Property 1 and rationalises the aggregate demand of the H consumers. When considering an economy of consumers with changing tastes, each consumer is characterised by the pair of utilities $U_h^{(1)}(c_{h,1}, c_{h,2}, c_{h,3})$ and $U_h^{(2)}(c_{h,2}, c_{h,3} | \; c_{h,1})$. Then given aggregate naive or sophisticated demands, does a non-changing tastes U_A exist which generates the sum of the demands for the naive or sophisticated consumers (where the U_A may be different for the two cases)?

We next provide a sufficient condition such that the naive and sophisticated demands generated by an economy of effectively consistent agents with changing tastes can be rationalised by a common aggregator utility denoted $\widehat{U}_A$ (the proof is provided in the online Appendix N). The resulting aggregate demands satisfy $\sum_{h=1}^{H} \mathbf{c}_h^{*} = \sum_{h=1}^{H} \mathbf{c}_h^{**}$ and the aggregator inherits the effectively consistent form of utility of the set of individual consumer utilities $\{\widehat{U}_1, ..., \widehat{U}_h, ..., \widehat{U}_H\}$.

Proposition 10. *In a three period heterogeneous exchange economy with H agents, the utilities of agent $h \in \{1, 2, ..., H\}$ are given by*

$$U_h^{(1)}(c_{h,1}, c_{h,2}, c_{h,3}) = f_h^{(1)}(g_h(c_{h,1})\, c_{h,2}, g_h(c_{h,1})\, c_{h,3}), \tag{25}$$

where

$$g_h(c_{h,1}) = a_h c_{h,1}^{b_h} \;\; (a_h, b_h > 0), \tag{26}$$

$U_h^{(2)}(c_{h,2}, c_{h,3})$ *is homothetic and both* $U_h^{(1)}(c_{h,1}, c_{h,2}, c_{h,3})$ *and* $U_h^{(2)}(c_{h,2}, c_{h,3})$ *satisfy Property 1.*[53] *If the endowments are collinear, i.e., eqn. (24) is satisfied, then the aggregate demands can be rationalised by the effectively consistent aggregator*

$$\widehat{U}_A(c_1, c_2, c_3) = g_A(c_1)\, u_A^{(2)}(c_2, c_3), \tag{27}$$

where $\widehat{U}_A(c_1, c_2, c_3)$ *satisfies Property 1,* $u_A^{(2)}(c_2, c_3)$ *is homogeneous of degree* 1 *and*

$$g_A(c_1) = c_1^b \qquad \text{and} \qquad b = \frac{\sum_{h=1}^{H} \frac{\omega_h}{1+\frac{1}{b_h}}}{1 - \sum_{h=1}^{H} \frac{\omega_h}{1+\frac{1}{b_h}}}.$$

Remark 5. *It should be noted that although the aggregate demands based on naive and sophisticated choice can be rationalised by a utility function if the conditions in Proposition 10 are satisfied, since* $U_h^{(1)}(c_{h,1}, c_{h,2}, c_{h,3})$ *need not be homothetic, there may not exist an aggregator for aggregate resolute demands.*

[53] It follows from the Corollary 1 that $g_h(c_{h,1})$ being a power function implies that, for each agent h, there exists a $\widehat{U}_h$ which is homothetic.

Given the specific form of the period one utility (25) - (26) in Proposition 10, what economic intuition can be given for the parameter b_h? To address this question, note that maximising eqns. (25) - (26) subject to (23) yields the following period one demand function

$$c_{h,1} = \frac{\omega_h(p_1\overline{c}_1 + p_2\overline{c}_2 + p_3\overline{c}_3)}{p_1(1 + \frac{1}{b_h})}, \tag{28}$$

where

$$\omega_h(p_1\overline{c}_1 + p_2\overline{c}_2 + p_3\overline{c}_3) = y_{h,1} = p_1c_{h,1} + p_2c_{h,2} + p_3c_{h,3} \tag{29}$$

corresponds to period one income for agent h. Combining eqns. (28) and (29) and solving for b_h yields

$$b_h = \frac{p_1c_{h,1}}{p_2c_{h,2} + p_3c_{h,3}},$$

implying that b_h is the ratio of the expenditure on period one consumption to the expenditure on period two and three consumption.

The process for finding the aggregator $\widehat{U}_A$ corresponding to (25) - (26) in Proposition 10 is illustrated by the following example (supporting calculations are provided in the online Appendix O).

Example 5. *Assume two consumers $h = 1, 2$ satisfying the effective consistency conditions in Proposition 4. For consumer 1,*

$$U_1^{(1)}(c_{1,1}, c_{1,2}, c_{1,3}) = \left(c_{1,1}^{\frac{1}{2}}c_{1,2}\right)^{\frac{1}{2}} + \left(c_{1,1}^{\frac{1}{2}}c_{1,3}\right)^{\frac{1}{4}} \quad \textit{and} \quad U_1^{(2)}(c_{1,2}, c_{1,3}) = \sqrt{c_{1,2}} + \sqrt{c_{1,3}}$$

$$\widehat{U}_1(c_{1,1}, c_{1,2}, c_{1,3}) = \sqrt{c_{1,1}}\left(\sqrt{c_{1,2}} + \sqrt{c_{1,3}}\right)^2$$

and for consumer 2,

$$U_2^{(1)}(c_{2,1}, c_{2,2}, c_{2,3}) = (c_{2,1}c_{2,2})^{\frac{1}{2}} + (c_{2,1}c_{2,3})^{\frac{1}{4}} \quad \textit{and} \quad U_2^{(2)}(c_{2,2}, c_{2,3}) = \ln c_{2,2} + \ln c_{2,3}$$

$$\widehat{U}_2(c_{2,1}, c_{2,2}, c_{2,3}) = c_{2,1}\sqrt{c_{2,2}c_{2,3}}.$$

The utilities $U_h^{(1)}$, $h = 1, 2$, take the form required by Proposition 10, where $g_h(c_{h,1}) = c_{h,1}^{b_h}$ and $b_1 = 1/2$ and $b_2 = 1$. Each consumer h considers the optimisation problems P_1 and P_2 with the modified budget constraint (23). Assume the special form of proportional endowments, $\overline{\mathbf{c}}_1 = \overline{\mathbf{c}}_2$. To construct $\widehat{U}_A$ using eqn. (27) in Proposition 10, it is first necessary to derive $u_A^{(2)}(c_2, c_3)$ and the b_A used in $g_A(c_1) = c_1^{b_A}$. After solving for the aggregate demand functions, one can follow the Hurwicz and Uzawa (1971) integration process to obtain $u_A^{(2)}(c_2, c_3)$. First, it can be verified that the corresponding indirect utility aggregator is

$$v_A^{(2)}(c_2, c_3) = \frac{m}{p^{\frac{11}{14}}(1+p)^{-\frac{4}{7}}},$$

where $p = p_2/p_3$ *and* $m = 7(p_1\bar{c}_1 + p_2\bar{c}_2 + p_3\bar{c}_3)/(12p_3)$ *are the normalised price and income, respectively. Second, the direct utility function* $u_A^{(2)}(c_2, c_3)$ *can be obtained by solving for the inverse demand functions* $p(c_2, c_3)$ *and* $m(c_2, c_3)$ *and then substituting them into* $v_A^{(2)}(c_2, c_3)$. *The function* $g_A(c_1)$ *takes the form* $g_A(c_1) = c_1^{b_A}$, *where*

$$\frac{1}{1+\frac{1}{b_A}} = \frac{1}{2}\frac{1}{1+\frac{1}{b_1}} + \frac{1}{2}\frac{1}{1+\frac{1}{b_2}} = \frac{1}{2(1+2)} + \frac{1}{2(1+1)},$$

implying that $g_A(c_1) = c_1^{\frac{5}{7}}$. *Since* $u_A^{(2)}(c_2, c_3)$ *is homogeneous of degree* 1, *it can be substituted into (27) yielding*

$$\widehat{U}_A(c_1, c_2, c_3) = \frac{c_1^{\frac{5}{7}}\left[\frac{3(c_3-c_2)+\sqrt{9c_2^2+9c_3^2+466c_2c_3}}{22} + c_3\right]}{\left[\frac{3(c_3-c_2)+\sqrt{9c_2^2+9c_3^2+466c_2c_3}}{22c_2}\right]^{\frac{11}{14}}\left[1 + \frac{3(c_3-c_2)+\sqrt{9c_2^2+9c_3^2+466c_2c_3}}{22c_2}\right]^{-\frac{4}{7}}}. \tag{30}$$

Thus the aggregate demands can be obtained by maximising the aggregator (30) subject to the aggregate version of the budget constraint (23).

For the case of heterogeneous quasilinear effectively consistent consumers, a similar aggregation result holds except that no restriction is required on the distribution of initial endowments. (See the online Appendix P.)

7. Concluding Comments

In this paper assuming changing tastes, we provide two different restrictions on preferences which are sufficient for plans to be effectively consistent. Revealed preference tests for the utility forms associated with effective consistency are derived and the changing tastes complications associated with Dutch Books, naive and sophisticated equilibria and aggregation are investigated. A number of open questions remain. First, are there other forms of utility beyond those specified in Propositions 4 and 6 which result in effectively consistent plans? Second, is our conjecture correct that naive and sophisticated choices must agree when either optimal plan can be rationalised? Third, is it possible to derive other conditions for aggregation when consumers exhibit changing tastes? Fourth, does the notion of effective consistency extend to uncertainty settings and how is this related to the literature on inconsistency associated with various forms of non-Expected Utility preferences?

Appendix

A. Proof of Proposition 4

Consider naive choice. Define $x = g(c_1)\,c_2$ and $y = g(c_1)\,c_3$. The first order conditions are given by

$$\frac{g(c_1)\,f_x^{(1)}}{g'(c_1)\left(f_x^{(1)}c_2 + f_y^{(1)}c_3\right)} = \frac{p_2}{p_1} \tag{A.1}$$

and

$$\frac{f_x^{(1)}}{f_y^{(1)}} = \frac{p_2}{p_3}, \tag{A.2}$$

where $f_x^{(1)} = \partial f^{(1)}/\partial x$ and $f_y^{(1)} = \partial f^{(1)}/\partial y$. Substituting eqn. (A.2) into eqn. (A.1), yields

$$\frac{g(c_1)}{g'(c_1)(p_2c_2 + p_3c_3)} = \frac{1}{p_1} \Leftrightarrow p_2c_2 + p_3c_3 = \frac{p_1 g(c_1)}{g'(c_1)},$$

implying that $p_1c_1 + p_1 g(c_1)/g'(c_1) = y_1$. Denote the solution to the above equation as c_1^*. Next consider sophisticated choice. The first order condition in period two is

$$\frac{f_x^{(2)}}{f_y^{(2)}} = \frac{p_2}{p_3}. \tag{A.3}$$

Combining the above equation with the budget constraint $y_1 - p_1c_1 \geq p_2c_2 + p_3c_3$, one can solve for c_2 and c_3 as functions of c_1. Since c_1^* is also the unique solution to the following optimisation problem

$$\max_{c_1,c_2,c_3} f^{(2)}(g(c_1)\,c_2, g(c_1)\,c_3) \quad S.T.\ y_1 \geq p_1c_1 + p_2c_2 + p_3c_3,$$

it follows that

$$f_x^{(2)}\left[g'(c_1)\,c_2 + \frac{\partial c_2}{\partial c_1}g(c_1)\right] + f_y^{(2)}\left[g'(c_1)\,c_3 + \frac{\partial c_3}{\partial c_1}g(c_1)\right]\Bigg|_{c_1=c_1^*} = 0.$$

Next we want to argue that

$$g'(c_1)\,c_2 + \frac{\partial c_2}{\partial c_1}g(c_1)\Bigg|_{c_1=c_1^*} = g'(c_1)\,c_3 + \frac{\partial c_3}{\partial c_1}g(c_1)\Bigg|_{c_1=c_1^*} = 0.$$

If that were not the case, when $c_1 = c_1^*$, it would follow that

$$-\frac{f_x^{(2)}}{f_y^{(2)}} = \frac{g'(c_1)\,c_3 + \frac{\partial c_3}{\partial c_1}g(c_1)}{g'(c_1)\,c_2 + \frac{\partial c_2}{\partial c_1}g(c_1)}.$$

Taking the derivative on both sides of (A.3) with respect to c_1 yields

$$\frac{f_{xx}^{(2)} - \frac{f_x^{(2)}}{f_y^{(2)}} f_{xy}^{(2)}}{\frac{f_x^{(2)}}{f_y^{(2)}} f_{yy}^{(2)} - f_{xy}^{(2)}} = \frac{g'(c_1)\, c_3 + \frac{\partial c_3}{\partial c_1} g(c_1)}{g'(c_1)\, c_2 + \frac{\partial c_2}{\partial c_1} g(c_1)}.$$

Therefore, when $c_1 = c_1^*$, we have

$$\frac{f_{xx}^{(2)} - \frac{f_x^{(2)}}{f_y^{(2)}} f_{xy}^{(2)}}{\frac{f_x^{(2)}}{f_y^{(2)}} f_{yy}^{(2)} - f_{xy}^{(2)}} = -\frac{f_x^{(2)}}{f_y^{(2)}},$$

or equivalently

$$\left(f_y^{(2)}\right)^2 f_{xx}^{(2)} + \left(f_x^{(2)}\right)^2 f_{yy}^{(2)} - 2 f_x^{(2)} f_y^{(2)} f_{xy}^{(2)} = 0,$$

which violates the assumption that $f^{(2)}$ is strictly quasiconcave. Therefore, there must exist a unique c_1^* such that

$$g'(c_1)\, c_2 + \frac{\partial c_2}{\partial c_1} g(c_1)\bigg|_{c_1 = c_1^*} = g'(c_1)\, c_3 + \frac{\partial c_3}{\partial c_1} g(c_1)\bigg|_{c_1 = c_1^*} = 0.$$

Hence

$$f_x^{(1)} \left[g'(c_1)\, c_2 + \frac{\partial c_2}{\partial c_1} g(c_1)\right] + f_y^{(1)} \left[g'(c_1)\, c_3 + \frac{\partial c_3}{\partial c_1} g(c_1)\right] = 0$$

if and only if $c_1 = c_1^*$, implying that c_1^* is also the unique solution for sophisticated choice. Thus there exists a unique $\mathbf{c}^{**}$ such that $\mathbf{c}^* = \mathbf{c}^{**}$. If one takes $\widehat{U}(c_1, c_2, c_3) = U^{(2)}(c_2, c_3)$, it can be easily verified that the common plan can be rationalised by $\widehat{U}$. Since $U^{(2)}$ satisfies Property 1, $\widehat{U}$ also satisfies Property 1.

B. Proof of Corollary 1

Effective consistency directly follows from Proposition 4. We only need to prove that $\widehat{U}(c_1, c_2, c_3) = g(c_1)\, u^{(2)}(c_2, c_3)$. The first order conditions yield

$$\frac{g'(c_1)\, u^{(2)}(c_2, c_3)}{g(c_1) \frac{\partial u^{(2)}(c_2, c_3)}{\partial c_2}} = \frac{p_1}{p_2} \Leftrightarrow c_2 \frac{\partial u^{(2)}(c_2, c_3)}{\partial c_2} \frac{p_1 g(c_1)}{g'(c_1)} = p_2 c_2 u^{(2)}(c_2, c_3) \qquad \text{(B.1)}$$

and

$$\frac{g'(c_1)\, u^{(2)}(c_2, c_3)}{g(c_1) \frac{\partial u^{(2)}(c_2, c_3)}{\partial c_3}} = \frac{p_1}{p_3} \Leftrightarrow c_3 \frac{\partial u^{(2)}(c_2, c_3)}{\partial c_3} \frac{p_1 g(c_1)}{g'(c_1)} = p_3 c_3 u^{(2)}(c_2, c_3). \qquad \text{(B.2)}$$

Since $u^{(2)}(c_2, c_3)$ is homogenous of degree 1, Euler's equation gives that

$$c_2 \frac{\partial u^{(2)}(c_2, c_3)}{\partial c_2} + c_3 \frac{\partial u^{(2)}(c_2, c_3)}{\partial c_3} = u^{(2)}(c_2, c_3). \tag{B.3}$$

Adding eqns. (B.1) and (B.2) together and using (B.3), one can obtain $p_1 g(c_1)/g'(c_1) = p_2 c_2 + p_3 c_3$. Substituting this expression into the budget constraint yields $p_1 c_1 + p_1 g(c_1)/g'(c_1) = y_1$. Therefore, maximising $\widehat{U}$ results in the same demands as naive and sophisticated choice.

C. Proof of Proposition 5

First since we assume that $U^{(2)}$ is homothetic, it follows from Blackorby, *et al.* (1973, Theorem 5) that sophisticated choice is unique. To simplify notation in this proof, (c_1, c_2, c_3) is used instead of $(c_1^{**}, c_2^{**}, c_3^{**})$ to denote the optimal demands from sophisticated choice. Since $U^{(2)}(c_2, c_3)$ is homothetic, following sophisticated choice results in $c_2 = \gamma_2(y_1 - p_1 c_1)$ and $c_3 = \gamma_3(y_1 - p_1 c_1)$, where $\gamma_2 > 0$ and $\gamma_3 > 0$ are functions of (p_2, p_3). It can be verified that

$$\frac{\partial c_1}{\partial p_2} + c_2 \frac{\partial c_1}{\partial y_1} = \frac{\partial c_1}{\partial p_2} + \gamma_2(y_1 - p_1 c_1)\frac{\partial c_1}{\partial y_1}$$

and

$$\frac{\partial c_2}{\partial p_1} + c_1 \frac{\partial c_2}{\partial y_1} = -p_1 \gamma_2 \left(\frac{\partial c_1}{\partial p_1} + c_1 \frac{\partial c_1}{\partial y_1}\right).$$

If the demands can be rationalised, the Slutsky matrix is symmetric, implying that $\partial c_1/\partial p_2 + c_2 \partial c_1/\partial y_1 = \partial c_2/\partial p_1 + c_1 \partial c_2/\partial y_1$, or equivalently

$$\frac{\partial c_1}{\partial p_2} + \gamma_2(y_1 - p_1 c_1)\frac{\partial c_1}{\partial y_1} = -p_1 \gamma_2 \left(\frac{\partial c_1}{\partial p_1} + c_1 \frac{\partial c_1}{\partial y_1}\right). \tag{C.1}$$

Noticing that the c_1 demand function is homogeneous of degree zero in prices and income, it follows from Euler's Theorem that

$$p_1 \frac{\partial c_1}{\partial p_1} + p_2 \frac{\partial c_1}{\partial p_2} + p_3 \frac{\partial c_1}{\partial p_3} + y_1 \frac{\partial c_1}{\partial y_1} = 0,$$

which is equivalent to

$$\frac{\partial c_1}{\partial p_1} = -\frac{1}{p_1}\left(p_2 \frac{\partial c_1}{\partial p_2} + p_3 \frac{\partial c_1}{\partial p_3} + y_1 \frac{\partial c_1}{\partial y_1}\right). \tag{C.2}$$

Substituting (C.2) into (C.1), results in

$$\frac{\partial c_1}{\partial p_2} = \gamma_2 \left(p_2 \frac{\partial c_1}{\partial p_2} + p_3 \frac{\partial c_1}{\partial p_3} \right). \tag{C.3}$$

Given that $p_2 c_2 + p_3 c_3 = (p_2 \gamma_2 + p_3 \gamma_3)(y_1 - p_1 c_1) = y_1 - p_1 c_1$, it follows that

$$p_2 \gamma_2 + p_3 \gamma_3 = 1.$$

Combining this equation with (C.3) yields $\gamma_3 \partial c_1 / \partial p_2 = \gamma_2 \partial c_1 / \partial p_3$, or equivalently $c_3^{**} \partial c_1^{**} / \partial p_2 = c_2^{**} \partial c_1^{**} / \partial p_3$.

D. Proof of Proposition 6

For naive choice, it can be verified that $\partial f^{(1)} / \partial c_1 = p_1 / p_3$ and $\partial g^{(2)} / \partial c_2 = p_2 / p_3$, which will be recognised to be the first order conditions associated with maximising $\widehat{U}(c_1, c_2, c_3) = f^{(1)}(c_1) + g^{(2)}(c_2) + c_3$. This implies that naive choice agrees with the optimal demands for $\widehat{U}$. For sophisticated choice, it follows from the first order condition associated with maximising $U^{(2)}$ that $\partial g^{(2)} / \partial c_2 = p_2 / p_3$, implying that c_2^{**} is independent of c_1^{**} and y_1. Therefore conditional on the optimal c_1, one must have

$$0 = \frac{\partial U^{(1)}}{\partial c_1} = \frac{\partial f^{(1)}}{\partial c_1} + \frac{\partial}{\partial c_1} \left(\frac{y_1 - p_1 c_1 - p_2 c_2}{p_3} \right) = \frac{\partial f^{(1)}}{\partial c_1} - \frac{p_1}{p_3},$$

and thus sophisticated choice also agrees with the optimal demands for $\widehat{U}$. Since $U^{(1)}$ and $U^{(2)}$ satisfy Property 1, clearly $\widehat{U}$ also satisfies Property 1. Thus maximising $\widehat{U}$ has a unique solution implying that optimal sophisticated demands are also unique.

E. Proof of Proposition 7

To see that (i) implies (ii), first compute the first order conditions

$$\left(f_x \left(g\left(c_1^i\right) c_2^i, g\left(c_1^i\right) c_3^i\right) c_2^i + f_y \left(g\left(c_1^i\right) c_2^i, g\left(c_1^i\right) c_3^i\right) c_3^i\right) g'\left(c_1^i\right) = \lambda^i p_1^i,$$

$$f_x \left(g\left(c_1^i\right) c_2^i, g\left(c_1^i\right) c_3^i\right) g\left(c_1^i\right) = \lambda^i p_2^i$$

and

$$f_y \left(g\left(c_1^i\right) c_2^i, g\left(c_1^i\right) c_3^i\right) g\left(c_1^i\right) = \lambda^i p_3^i,$$

which imply

$$f_x \left(g\left(c_1^i\right) c_2^i, g\left(c_1^i\right) c_3^i\right) = \frac{\lambda^i p_2^i}{g\left(c_1^i\right)} \tag{E.1}$$

$$f_y\left(g\left(c_1^i\right)c_2^i, g\left(c_1^i\right)c_3^i\right) = \frac{\lambda^i p_3^i}{g\left(c_1^i\right)} \tag{E.2}$$

and

$$g'\left(c_1^i\right) = \frac{\lambda^i p_1^i}{f_x\left(g\left(c_1^i\right)c_2^i, g\left(c_1^i\right)c_3^i\right)c_2^i + f_y\left(g\left(c_1^i\right)c_2^i, g\left(c_1^i\right)c_3^i\right)c_3^i} = \frac{g\left(c_1^i\right)p_1^i}{p_2^i c_2^i + p_3^i c_3^i}. \tag{E.3}$$

Defining $G^i = g\left(c_1^i\right)$ $(i = 1, 2, ..., N)$, since f and g are both strictly increasing, we have $\left(G^i\right)_{i=1}^N > 0$ and $\left(\lambda^i\right)_{i=1}^N > 0$. Because f and g are both concave,

$$\begin{aligned} & f\left(g\left(c_1^i\right)c_2^i, g\left(c_1^i\right)c_3^i\right) \\ \leq\ & f\left(g\left(c_1^j\right)c_2^j, g\left(c_1^j\right)c_3^j\right) + \\ & f_x\left(g\left(c_1^j\right)c_2^j, g\left(c_1^j\right)c_3^j\right)\left[g\left(c_1^i\right)c_2^i - g\left(c_1^j\right)c_2^j\right] + \\ & f_y\left(g\left(c_1^j\right)c_2^j, g\left(c_1^j\right)c_3^j\right)\left[g\left(c_1^i\right)c_3^i - g\left(c_1^j\right)c_3^j\right] \end{aligned}$$

and

$$g\left(c_1^i\right) \leq g\left(c_1^j\right) + g'\left(c_1^j\right)\left(c_1^i - c_1^j\right).$$

Substituting eqns. (E.1) - (E.3) into the above two inequalities and denoting $F^i = f\left(g\left(c_1^i\right)c_2^i, g\left(c_1^i\right)c_3^i\right)$ and $G^i = g\left(c_1^i\right)$ $(i = 1, 2, ..., N)$ yields

$$F^i \leq F^j + \lambda^j p_2^j\left(\frac{G^i}{G^j}c_2^i - c_2^j\right) + \lambda^j p_3^j\left(\frac{G^i}{G^j}c_3^i - c_3^j\right) \tag{E.4}$$

and

$$G^i \leq G^j\left[1 + \frac{p_1^j\left(c_1^i - c_1^j\right)}{p_2^j c_2^j + p_3^j c_3^j}\right]. \tag{E.5}$$

To see that (ii) implies (i), assume that there exist real numbers $\left(F^i\right)_{i=1}^N$, $\left(G^i\right)_{i=1}^N > 0$ and $\left(\lambda^i\right)_{i=1}^N > 0$ such that the inequalities (E.4) and (E.5) hold. Then one can assume the following utility functions

$$g\left(c_1\right) = \min_j\left(G^j\left[1 + \frac{p_1^j\left(c_1 - c_1^j\right)}{p_2^j c_2^j + p_3^j c_3^j}\right]\right) \tag{E.6}$$

and

$$f\left(g\left(c_1\right)c_2, g\left(c_1\right)c_3\right) = \min_i\left(F^i + \lambda^i p_2^i\left[\frac{g\left(c_1\right)}{G^i}c_2 - c_2^i\right] + \lambda^i p_3^i\left[\frac{g\left(c_1\right)}{G^i}c_3 - c_3^i\right]\right). \tag{E.7}$$

Note that $\left(G^i\right)_{i=1}^N > 0$ and $\left(\lambda^i\right)_{i=1}^N > 0$, $g\left(c_1\right)$ and $f\left(g\left(c_1\right)c_2, g\left(c_1\right)c_3\right)$ are strictly increasing. Define $x = g\left(c_1\right)c_2$ and $y = g\left(c_1\right)c_3$. Since the utility function (E.6) is piecewise linear in c_1 and the utility function (E.7) is piecewise linear in (x, y), it is straightforward to show that they are non-satiated and concave in c_1 and (x, y), respectively. Next we want to argue the utility function (E.7) can rationalise the data. It is straightforward to verify that $f\left(g\left(c_1^j\right)c_2^j, g\left(c_1^j\right)c_3^j\right) = F^j$ for all $j = 1, 2, ..., N$. Now suppose that

$$p_1^l c_1^l + p_2^l c_2^l + p_3^l c_3^l \geq p_1^l c_1 + p_2^l c_2 + p_3^l c_3.$$

Then

$$\begin{aligned}
& f\left(g\left(c_1\right)c_2, g\left(c_1\right)c_3\right) \\
= & \min_i \left(\begin{array}{c} F^i + \lambda^i p_2^i \left\{ \dfrac{\min_j \left(G^j \left[1 + \frac{p_1^j\left(c_1 - c_1^j\right)}{p_2^j c_2^j + p_3^j c_3^j}\right]\right)}{G^i} c_2 - c_2^i \right\} \\ + \lambda^i p_3^i \left\{ \dfrac{\min_j \left(G^j \left[1 + \frac{p_1^j\left(c_1 - c_1^j\right)}{p_2^j c_2^j + p_3^j c_3^j}\right]\right)}{G^i} c_3 - c_3^i \right\} \end{array} \right)
\end{aligned}$$

$$\begin{aligned}
\leq & \; F^l + \lambda^l p_2^l \left\{ \frac{G^l \left[1 + \frac{p_1^l\left(c_1 - c_1^l\right)}{p_2^l c_2^l + p_3^l c_3^l}\right]}{G^l} c_2 - c_2^l \right\} + \lambda^l p_3^l \left\{ \frac{G^l \left[1 + \frac{p_1^l\left(c_1 - c_1^l\right)}{p_2^l c_2^l + p_3^l c_3^l}\right]}{G^l} c_3 - c_3^l \right\} \\
= & \; F^l + \lambda^l \frac{p_1^l\left(c_1 - c_1^l\right)\left(p_2^l c_2 + p_3^l c_3\right) + \left[p_2^l\left(c_2 - c_2^l\right) + p_3^l\left(c_3 - c_3^l\right)\right]\left(p_2^l c_2^l + p_3^l c_3^l\right)}{p_2^l c_2^l + p_3^l c_3^l} \\
= & \; F^l + \lambda^l \left[p_1^l\left(c_1 - c_1^l\right) + p_2^l\left(c_2 - c_2^l\right) + p_3^l\left(c_3 - c_3^l\right)\right] \\
& + \frac{\lambda^l p_1^l\left(c_1 - c_1^l\right)\left[p_2^l\left(c_2 - c_2^l\right) + p_3^l\left(c_3 - c_3^l\right)\right]}{p_2^l c_2^l + p_3^l c_3^l}.
\end{aligned}$$

If $p_1^l c_1^l + p_2^l c_2^l + p_3^l c_3^l = p_1^l c_1 + p_2^l c_2 + p_3^l c_3$, then

$$p_1^l\left(c_1 - c_1^l\right) = -\left[p_2^l\left(c_2 - c_2^l\right) + p_3^l\left(c_3 - c_3^l\right)\right],$$

implying that

$$p_1^l\left(c_1 - c_1^l\right)\left[p_2^l\left(c_2 - c_2^l\right) + p_3^l\left(c_3 - c_3^l\right)\right] = -\left[p_1^l\left(c_1 - c_1^l\right)\right]^2 \leq 0$$

and hence

$$\begin{aligned}
f\left(g\left(c_1\right)c_2, g\left(c_1\right)c_3\right) & \leq F^l + \lambda^l \left[p_1^l\left(c_1 - c_1^l\right) + p_2^l\left(c_2 - c_2^l\right) + p_3^l\left(c_3 - c_3^l\right)\right] \\
& = F^l = f\left(g\left(c_1^l\right)c_2^l, g\left(c_1^l\right)c_3^l\right).
\end{aligned}$$

If $p_1^l c_1^l + p_2^l c_2^l + p_3^l c_3^l > p_1^l c_1 + p_2^l c_2 + p_3^l c_3$, then one can always find a (c_1, c_2, c_3^0) such that $p_1^l c_1^l + p_2^l c_2^l + p_3^l c_3^l = p_1^l c_1 + p_2^l c_2 + p_3^l c_3^0$. Since $c_3^0 > c_3$ and $f(g(c_1)c_2, g(c_1)c_3)$ is a strictly increasing function, we have

$$\begin{aligned} f(g(c_1)c_2, g(c_1)c_3) &< f\left(g(c_1)c_2, g(c_1)c_3^0\right) \\ &\leq F^l + \lambda^l\left[p_1^l\left(c_1 - c_1^l\right) + p_2^l\left(c_2 - c_2^l\right) + p_3^l\left(c_3^0 - c_3^l\right)\right] \\ &= F^l = f\left(g\left(c_1^l\right)c_2^l, g\left(c_1^l\right)c_3^l\right). \end{aligned}$$

Therefore $f(g(c_1)c_2, g(c_1)c_3) \leq f\left(g\left(c_1^l\right)c_2^l, g\left(c_1^l\right)c_3^l\right)$ always holds. Finally to see that (ii) and (iii) are equivalent, note that the inequality (E.4) can be rewritten as

$$F^i \leq F^j + \lambda^j \frac{p_2^j}{G^j}\left(G^i c_2^i - G^j c_2^j\right) + \lambda^j \frac{p_3^j}{G^j}\left(G^i c_3^i - G^j c_3^j\right),$$

which can be viewed as the traditional Afriat (1967) inequality corresponding to the demands $(G^i c_2^i, G^i c_3^i)$ and prices $(p_2^i/G^i, p_3^i/G^i)$. Since the traditional Afriat inequality is equivalent to GARP, inequality (E.4) is equivalent to $(G^i c_2^i, G^i c_3^i; p_2^i/G^i, p_3^i/G^i)$ satisfying GARP.

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