A NEW REPRESENTATION OF PREFERENCES OVER "CERTAIN × UNCERTAIN" CONSUMPTION PAIRS: THE "ORDINAL CERTAINTY EQUIVALENT" HYPOTHESIS

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For problems involving choices over "certain × uncertain" consumption pairs, it is almost universally assumed that the decision maker's preferences can be represented by an expected TPC (two-period cardinal) utility function. In this paper, we present an alternative representation of preferences, referred to as the "ordinal certainty equivalent" hypothesis, which we argue (i) is at least as intuitive as the expected utility hypothesis, (ii) includes the corresponding TPC representation as a special case with the set of cases not expressible in the latter format being both large and important, and (iii) is based on a more sensible hypothesis concerning the connection between "risk" and "time" preferences.

1. INTRODUCTION AND SUMMARY

RESTRICTING OUR ATTENTION to choices over "certain × uncertain" consumption pairs, we present in this paper an alternative to the *multiattribute* expected utility hypothesis. Assume that for each value of first-period consumption a DM (decision maker) possesses conditional "risk" preferences for uncertain second-period consumption which can be represented by a (single-attribute) second-period expected utility function. Then the *intertemporal* choice among pairs of certain first-period consumption and cumulative distribution functions on second-period consumption can be decomposed into two steps. First, these pairs can be converted into certain first-period and certainty equivalent second-period consumption pairs by using the DM's period-two expected utility function. Then the latter pairs can be ordered by an *ordinal* time preference relation defined on certain consumption plans. Theorem 1 ensures that such a function exists. Together these two steps are order-preserving and will be referred to as the *OCE* (*ordinal certainty equivalent*) representation hypothesis. Thus our proposed alternative to the *two-period* (*multiattribute*) expected utility model is based on a conditional *second-period* (*single-attribute*) expected utility function and a *two-period* ordinal index.

This new OCE representation hypothesis is shown to include the corresponding TPC (two-period cardinal) expected utility paradigm as a quite limited special case. The latter requires additional axiomatic structure, a "coherence"

axiom (cf., Rossman and Selden [23]), which results in a specific, strong inter-dependence between risk and time preferences. In contrast, the more general OCE representation hypothesis permits one to *prescribe* risk and time pref-erences separately—thereby making possible an explicit *modelling* of their interrelationship.

The fact that a distinction can be made between attitudes toward risk and ordinal preferences over commodity bundles was noted by Kihlstrom and Mirman [14] within the context of a multiattribute expected utility model.[2] However, the OCE utility hypothesis enables one to go significantly further in distinguishing between the *effects* of risk and time preferences on choices among "certain × uncertain" consumption plans and hence may yield a number of new behavioral insights. Thus, for instance, the OCE representation is shown in [31] to shed considerable light on the effect of an increase in "capital risk" on thrift and in [29] to provide insight into the uncertainty generalization of the classic Fisherian equality between the "marginal rate of time preference" and the sure rate of interest.

In the next section we introduce notation and a number of important definitions. The principal representation theorem is presented in Section 3. Section 4 contains a brief geometric analysis and 5 examines the relationship between the TPC and OCE representations and presents a brief example illus-trating the potential applicability of the latter to specific economic problems. In the final section we speculate on generalizations and extensions.

2. NOTATION AND DEFINITIONS
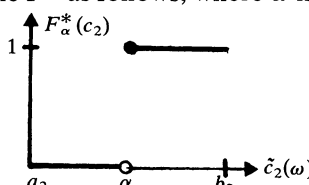
We shall employ the following notation and elementary definitions:

c_t:	value of real, generalized consumption flow in time-period t $(= 1, 2)$;
$C_t = [a_t, b_t]$:	set of all c_t-values, i.e., $c_t \in C_t$;
$C =_{\text{def}} C_1 \times C_2$:	certainty consumption (product) set;
$\overset{C}{\leqslant}$:	DM's (decision maker's) complete preorder-ing[3] on C;
$\omega \in \Omega$:	particular state in abstract states of nature space;
$\tilde{c}_2: \Omega \to C_2 \subset \mathbb{R}$:	random variable mapping the states of nature space into the real line, the value of which denotes the DM's real, generalized consump-tion flow for time-period two;
$F, G: [a_2, b_2] \to [0, 1]$:	(cumulative) distribution functions of the random variables $\tilde{c}_2^F, \tilde{c}_2^G$, respectively;
$\bar{X}$:	set of monotone non-decreasing, right continuous functions such that $F(b_2) = 1$ and $F(a_2) = 0$ (i.e., on $[a_2, b_2]$);

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[2] Cf., Selden [31, footnote 39].
[3] Cf., Debreu [2, p. 110].

$\overset{\bar{X}}{\lesssim}$: DM's complete preference preordering defined on the distribution function space;

F^*, G^*: one-point cumulative distribution functions. As an example define F^* as follows, where α is the "saltus point":

$$F_\alpha^* = \begin{cases} 0, & \tilde{c}_2(\omega) < \alpha, \\ 1, & \tilde{c}_2(\omega) \geqslant \alpha; \end{cases}$$

$\underline{\bar{X}}^* \subset \underline{\bar{X}}$: set of one-point cumulative distribution functions;

$V_{c_1} : (\tilde{c}_2(\omega)) \mapsto$ DM's time-period two conditional, cardinal
$V_{c_1}(\tilde{c}_2(\omega)) \in \mathbb{R}$: (von Neumann-Morgenstern) utility function;

$S =_{\text{def}} C_1 \times \bar{X}$: space of (c_1, F)-pairs;

$S_{c_i'} =_{\text{def}} \{(c_1, F) | c_1 = c_1'$ the c_1'-cross-section.
and $F \in \bar{X}\} \subset S$:

$\overset{S}{\lesssim}$: DM's complete preference preordering on the product space S;

$S^* =_{\text{def}} C_1 \times \bar{X}^*$: space of (c_1, F^*)-pairs;

$\overset{S^*}{\lesssim}$: DM's complete preference preordering on the product space S^*.

In what follows, extensive use will be made of the notion of "conditional" (or in terms of the graphical presentation of Section 4, "cross-sectional") preferences. If one assumes that there exists a complete preference preordering over all of S, then there will exist a complete preordering on each subset $S_{c_1'}$. Each such ordering, denoted $\overset{S_{c_1'}}{\lesssim}$, is "conditional" in the sense that it describes the DM's preferences over the set of c_2-c.d.f.'s, $\bar{X}$, given that his first-period consumption flow is c_1'. Since in the present setting all of the uncertainty concerning one's consumption opportunities is restricted to time-period two, the set of relations $\{\overset{S_{c_1'}}{\lesssim} | c_1 \in C_1\}$ will be referred to as his conditional "risk" preferences. In general, one would expect that for some unequal c_1' and c_1'' and some $F, G \in \bar{X}$, $F \overset{S_{c_1'}}{<} G$ and $G \overset{S_{c_1''}}{<} F$ might be possible. In this case, $\bar{X}$ will be said to be "risk preference dependent" on C_1.

3. THE OCE REPRESENTATION HYPOTHESIS

The fundamental question of this paper may be posed as follows: Supposing that one possesses a complete preference preordering on the set of possible "certain $\times$ uncertain" consumption pairs S, then how can (and/or should) it be represented numerically in a simple and intuitive fashion? We shall propose a new answer to this question.

We shall employ the following assumptions:

ASSUMPTION 1: $\bar{X}$ *is the set of c.d.f.'s corresponding to some (topological)* *subspace of the space of all (countably additive) probability measures* $M(C_2)$, *which is endowed with the topology of weak convergence.* $\bar{X}$ *is a mixture space and the set of one-point c.d.f.'s (supported by the domain of* $\bar{X}$), $\bar{X}^*$, *is a subset thereof.*[4]

ASSUMPTION 2: *There exists a complete preference preordering on* S, $\overset{s}{\leqslant}$.

ASSUMPTION 3: $\overset{s}{\leqslant}$ *is representable by a continuous "Bernoulli index"*[5] Ψ: $S \to \mathbb{R}$.

If, for some c_1, $c_1' \in C_1$ and F, $G \in \bar{X}$, $(c_1, F) \overset{s}{\leqslant} (c_1', G)$, then Assumption 3 says that $\Psi(c_1, F) \leqslant \Psi(c_1', G)$. It clearly is silent on the specific form Ψ will take.

ASSUMPTION 4: *Each conditional preference ordering* $\overset{s_{c_1}}{\leqslant}$ *is "NM represent-able"*[6] *with the corresponding continuous "NM index"* V_{c_1} *being strictly mono-tonically increasing.*

ASSUMPTION 5: *The period-two "NM index"* $V_{c_1}(c_2) = V(c_1, c_2)$ *is continuous on* $C_1 \times C_2$.

Assumptions 4 and 5 must be interpreted with the greatest of care. Even though the *period-two* NM index is defined on $C_1 \times C_2$, Assumption 4 *ONLY justifies* using the expected utility principle for choices between points in S *characterized by a common value of first-period consumption*. Thus, for instance, suppose[7]

$$(1) \qquad V_{c_1}(c_2) = \log(c_2 - b_1 c_1).$$

The temptation is great to interpret this expression as a TPC index. However, this would be *totally unwarranted on the basis of Assumption 4*. It follows from

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[4] $M(C_2)$ is defined on the measurable space $(C_2, B(C_2))$ where C_2 is clearly a metric space and $B(C_2)$ is its Borel σ-field. See [**7**, p. 46]. For a discussion of the weak topology and the corresponding notion of convergence, see [**17**, p. 39ff]. Finally, the term "mixture space" is defined in Fishburn [**4**, p. 110].

[5] We use the term "Bernoulli index" to refer to any real-valued order-preserving representation where the completely preordered space is at least *partially stochastic*. Instead of *assuming* the existence of Ψ, one could prove it by placing topological restrictions on $\bar{X}$ and conditions on $\overset{s}{\leqslant}$ following Theorem 1 in [**7**, p. 47].

[6] $\overset{s_{c_1}}{\leqslant}$ is "NM representable" if there exists a continuous "Bernoulli index" $\Lambda_{c_1}: \bar{X} \to \mathbb{R}$ which is order-preserving and a continuous "NM index" V_{c_1} defined by

$$\Lambda_{c_1}(F) = \int_{C_2} V_{c_1}(c_2)\, dF(c_2).$$

[7] $c_2 - bc_1 > 0$.

Rossman and Selden [23] that Assumptions 1–5 are *not sufficient* to establish the *existence* of a TPC representation. (We shall return to this point in Section 5.) The correct way to interpret equation (1) and more generally Assumption 5 may be expressed as follows: the expected utility which a DM associates with a given c.d.f. for second-period consumption depends continuously on the level of his previous consumption, c_1. Thus V is a conditional (upon c_1) *single-attribute* NM index.

Related to this point, it is necessary to introduce some additional notation. The DM's certainty equivalent period-two consumption (associated with the random variable $\tilde{c}_2^F$) is defined by the equality

$$V_{c_1}(\hat{c}_2) = \int_{C_2} V_{c_1}(c_2)\, dF(c_2).$$

Given Assumption 4, let us introduce the following:

$$(2) \qquad \hat{c}_2(c_1, F) = V_{c_1}^{-1} \int_{C_2} V_{c_1}(c_2)\, dF(c_2).$$

We are now ready to state our key result.

THEOREM 1 (OCE Representation Theorem): *If Assumptions 1–5 hold, then there exist* (i) $\forall c_1 \in C_1$ *and* $F \in \bar{X}$, *a unique* $\hat{c}_2(c_1, F) \in \mathbb{R}$ *and* (ii) *a continuous function* $U: C_1 \times C_2 \to \mathbb{R}$ *which together represent* $\overset{s}{\leqslant}$ *in that*

$$(c_1, F) \overset{s}{\leqslant} (c_1', G) \Leftrightarrow U(c_1, \hat{c}_2(c_1, F)) \leqslant U(c_1', \hat{c}_2(c_1', G))$$

$\forall c_1, c_1' \in C_1$ *and* $F, G \in \bar{X}$.

PROOF: We shall proceed in a series of steps. *Step* (*a*). As suggested in footnote 6, $\overset{S_{c_1}}{\leqslant}$ being "NM representable" means that with respect to a *specific cross-section* S_{c_1},

$$F \overset{S_{c_1}}{\leqslant} G \text{ iff } \Lambda_{c_1}(F) = \int_{C_2} V_{c_1}(c_2)\, dF(c_2) \leqslant \Lambda_{c_1}(G) = \int_{C_2} V_{c_1}(c_2)\, dG(c_2).$$

Step (*b*). Since V_{c_1} is continuous (Assumption 4) and defined on $[a_2, b_2]$ and $E[V_{c_1}(\tilde{c}_2^F)] \in V_{c_1}[a_2, b_2]$, $\forall F \in \bar{X}$, it follows from the second Mean Value Theorem (for integrals) that $\exists$ a one-point c.d.f. $F^* \in \bar{X}$ such that $F \overset{S_{c_1}}{\sim} F^*$. Now if $\bar{X}$ is risk preference dependent on C_1, F^* will not generally be indifferent to F on a different cross-section, $S_{c_1'}$. *Step* (*c*). It is straightforward to show that since V_{c_1} is strictly monotonically increasing (Assumption 4) then on $[a_2, b_2]$, the $F^* \in \bar{X}$ indifferent to F (for a *fixed* c_1) is *unique*. *Step* (*d*). Let us continue to focus on the single cross-section S_{c_1}. Given that $\exists! F^*$ associated with each $F \in \bar{X}$

and each one-point c.d.f. by definition has a single saltus, then associated with each F will be a unique *period-two certainty equivalent* defined by $V_{c_1}(\hat{c}_2) = \Lambda_{c_1}(F^*) = \Lambda_{c_1}(F)$. *Step (e)*. Consider next two different cross sections, S_{c_1} and $S_{c'_1}$. By the preceding steps, we have that there exists a unique $F^*_{c_1} \overset{S_{c_1}}{\sim} F$ and a unique $F^*_{c'_1} \overset{S_{c'_1}}{\sim} F$ where, in general, $F^*_{c_1}$ and $F^*_{c'_1}$ are different. It follows that the corresponding period-two certainty equivalents are different. Thus, for each c.d.f. and value of first-period consumption there will exist a unique second-period certainty equivalent, denoted $\hat{c}_2(c_1, F)$. *Step (f)*. We next want to show that the certainty equivalent is continuous in c_1. First rewrite equation (2) as follows:

$$V(c_1, \hat{c}_2(c_1, F)) = \int_{C_2} V(c_1, c_2) \, dF(c_2).$$

Since this equality holds $\forall c_1 \in C_1$ and $F \in \bar{X}$, and since the RHS is continuous in c_1 via Assumption 5, then so must be the LHS. Suppose that $\hat{c}_2$ is not continuous in c_1, then V would not be continuous in c_1, but that is a contradiction. *Step (g)*. Let the map $l: C_1 \times C_2 \to C_1 \times \bar{X}^*$ be defined by $l: (c_1, c_2) \mapsto (c_1, F^*)$ where F^* has its jump at $c_2 \in C_2$. Then the ordinal index $U: C_1 \times C_2 \to \mathbb{R}$ can be obtained from the two-argument "Bernoulli index" Ψ via the following relationship:

$$U(c_1, c_2) = \Psi(l(c_1, c_2)) = \Psi(c_1, F^*).$$

The interrelationship among these three maps, U, Ψ, and l, is summarized by the following diagram:

$$
\begin{array}{ccc}
C_1 \times C_2 & \xrightarrow{\quad l \quad} & C_1 \times \bar{X}^* \\
 & \searrow U & \downarrow \Psi \\
 & & \mathbb{R}
\end{array}
$$
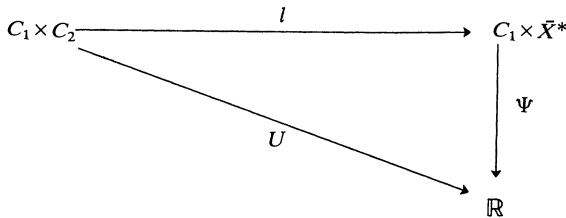
where U is defined so as to make the diagram "commute".

Having verified the existence claims (i) and (ii) of Theorem 1, we next want to show that the second-period certainty equivalent and U do represent preferences. *Step (h)*. Let us establish sufficiency,

$$(c_1, F) \overset{s}{\precsim} (c'_1, G) \Rightarrow U(c_1, \hat{c}_2(c_1, F)) \leqslant U(c'_1, \hat{c}_2(c'_1, G)).$$

Consider the points $(c'_1, F) \in S_{c'_1}$ and $(c''_1, G) \in S_{c''_1}$. Assumption 3 implies that

$$(3) \qquad (c'_1, F) \overset{s}{\precsim} (c''_1, G) \Leftrightarrow \Psi(c'_1, F) \leqslant \Psi(c''_1, G).$$

Step (a)–(e)$\Rightarrow \forall F \in \bar{X}$ on the *given* cross-section S_{c_i} $\exists! F^*$ such that $F^* \overset{S_{c_i}}{\sim} F$.

Then

$$F^* \overset{S_{c_i}}{\sim} F \Rightarrow (c_1', F^*) \overset{S}{\sim} (c_1', F) \quad \text{and}$$

$$G^* \overset{S_{c_i''}}{\sim} G \Rightarrow (c_1'', G^*) \overset{S}{\sim} (c_1'', G).$$

Since $\overset{S}{\leqslant}$ is "Bernoulli representable" by Assumption 3, we have

$$(c_1', F^*) \overset{S}{\sim} (c_1', F) \Rightarrow \Psi(c_1', F^*) = \Psi(c_1', F),$$

$$(c_1'', G^*) \overset{S}{\sim} (c_1'', G) \Rightarrow \Psi(c_1'', G^*) = \Psi(c_1'', G).$$

Substituting from the RHS of these latter expressions into the RHS of equation (3) above yields

$$(c_1', F) \overset{S}{\leqslant} (c_1'', G) \Leftrightarrow \Psi(c_1', F^*) \leqslant \Psi(c_1'', G^*).$$

But now since by steps (d) and (g)

$$\Psi(c_1', F^*) = U(c_1', \hat{c}_2(c_1', F)),$$
$$\Psi(c_1'', G^*) = U(c_1'', \hat{c}_2(c_1'', G)),$$

we have sufficiency upon substitution. *Step (i)*. Next consider necessity,

$$U(c_1, \hat{c}_2(c_1, F)) \leqslant U(c_1', \hat{c}_2(c_1', G)) \Rightarrow (c_1, F) \overset{S}{\leqslant} (c_1', G).$$

By step (d) and the definition of U in step (g), we have

$$U(c_1', \hat{c}_2(c_1', F)) \leqslant U(c_1'', \hat{c}_2(c_1'', G)) \Rightarrow \Psi(c_1', F^*) \leqslant \Psi(c_1'', G^*)$$

which, because $\overset{S}{\leqslant}$ is "Bernoulli representable" (Assumption 3), becomes

$$(c_1', F^*) \overset{S}{\leqslant} (c_1'', G^*).$$

Then simply reverse the reasoning in Step (h). This establishes necessity. *Step (j)*. Since the composition of two continuous functions is continuous, $U(= \Psi \circ l)$ will be continuous if Ψ and l are. Since Ψ is continuous by Assumption 3, we need only establish the continuity of l. But given that $\bar{X}$ is endowed with the weak topology, it is clear from Lemma 6.1 of Parthasarathy [17, p. 42] that l will be continuous. *Q.E.D.*

Together this theorem and its proof suggest that the choice between any two points in S, say (c_1, F) and (c_1', G), can be decomposed into fundamentally two steps. First, based on the DM's "risk" preferences, convert both points into equivalent certain first-period and certainty equivalent second-period consumption pairs according to

$$\hat{c}_2(c_1, F) = V_{c_1}^{-1} \int_{C_2} V_{c_1}(c_2) \, dF(c_2) \quad \text{and}$$

$$\hat{c}_2(c_1', G) = V_{c_1'}^{-1} \int_{C_2} V_{c_1'}(c_2) \, dG(c_2).$$

Then apply to the resulting pairs $(c_1, \hat{c}_2(c_1, F))$ and $(c_1', \hat{c}_2(c_1', G))$ the utility function U defined on "certain" consumption pairs which Theorem 1 ensures will exist. Together these two steps are order-preserving. Finally, it should be noted (as can be seen from the "necessity" portion of the proof) that corresponding to each V, U-pair will be a $\overset{S}{\leqslant}$ which is complete and a preordering.

The procedure described above is referred to as the "ordinal certainty equivalent" representation because of (1) the invariance of U under increasing monotonic transforms (cf., Corollary 1 below) and (2) the important role played by certainty equivalents. Our terminology is not meant to imply that the certainty equivalents are ordinal; rather, it seeks to emphasize the ordinality of U.

As one might expect, this OCE *representation* is not unique.

COROLLARY 1: *The OCE representation (Theorem 1) is unique up to an increasing monotonic transform of U and up to a positive affine transform of each V_{c_1}.*

This result can readily be verified by simple computation and hence its proof is omitted.

We conclude this section by considering an important special case.

DEFINITION 1: $\bar{X}$ will be said to be *"risk preference independent"* (r.p.i.) of C_1 iff $\forall c_1', c_1'' \in C_1$ and $F, G \in \bar{X}$

$$F \overset{S_{c_1'}}{\leqslant} G \Rightarrow F \overset{S_{c_1''}}{\leqslant} G.$$

What this says is that the DM's "risk preferences" for uncertain period-two consumption are independent of his level of first-period consumption.[8] Said another way, his degree of risk aversion[9] is not dependent on his previous

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[8] Decision theorists frequently assume a version of risk preference independence (often referred to as "utility independence") principally on the grounds of ease of implementation (e.g., see Fishburn [5]). In economics it is common to assume a form of utility which exhibits risk preference independence because of the tractability it lends to the analysis (e.g., see Samuelson [25] and Rubinstein [24]).

[9] E.g., in terms of the Arrow-Pratt measures of absolute or relative risk aversion—see [1 and 20]—where V is the relevant one-attribute "NM" index.

consumption experience. We can state the following result as an immediate consequence of Theorem 1:

COROLLARY 2: *If Assumptions 1–4 hold and $\bar{X}$ is r.p.i. of C_1, then the ordering $\overset{S}{\leqslant}$ will be OCE representable in that $\forall c_1, c_1' \in C_1$ and $F, G \in \bar{X}$,*

$$(c_1, F) \overset{S}{\leqslant} (c_1', G) \Leftrightarrow U(c_1, \hat{c}_2(F)) \leqslant U(c_1', \hat{c}_2(G)),$$

where

$$\hat{c}_2(F) = V^{-1} \int_{C_2} V(c_2) \, dF(c_2) \quad and \quad \hat{c}_2(G) = V^{-1} \int_{C_2} V(c_2) \, dG(c_2).$$

Risk preference independence greatly simplifies matters in that the same conditional period-two "NM index" V can be used despite the fact that F and G are associated with different first-period consumption flows.[10]

4. GRAPHICAL EXPOSITION

It is our contention that despite the seemingly technical nature of the OCE representation theorem just developed, the underlying ideas are quite simple. The purpose of this section is to provide a brief graphical summary of the more important of these ideas.

We find the geometric portrayal of the "certain × uncertain" choice space, S, in Figure 1 to be especially useful as an expository device.[11] One may view S as
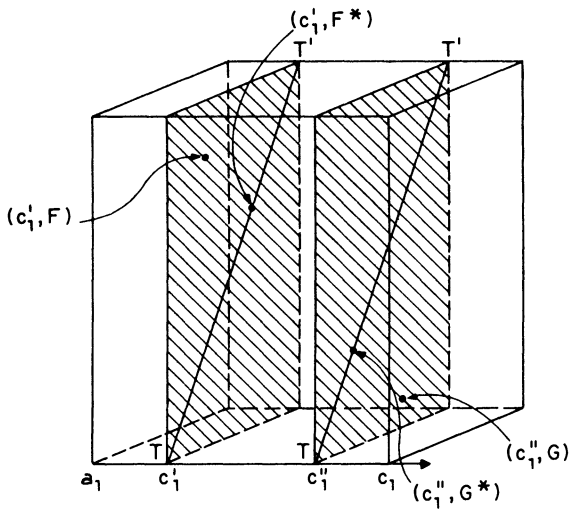

FIGURE 1

[10] Risk preference independence implies equivalence of each of the conditional orderings $\overset{S_{c_1}}{\leqslant}$. But this implies that the corresponding NM indices can differ at most by a positive affine transform.
[11] It is highly idealized and should not be interpreted literally.

being a "box" comprised of a series of "cross-sections" ordered by c_1 values beginning on the left side at $c_1 = a_1$ and ending on the right side at $c_1 = b_1$. Two specific cross-sections, $S_{c_1'}$ and $S_{c_1''}$, have been drawn. The set of points on a given S_{c_1} corresponds to the set of c.d.f.'s $\bar{X}$.[12] Each cross-section represents an identical copy of $\bar{X}$, and hence differs only in its respective c_1-value. The complete preference preordering $\overset{S}{\leqslant}$ lives on the entire "box".

Assumption 4 postulates that each cross-section is "NM representable" which implies that the indifference curves (constant expected utility) on each S_{c_1} are "linear" in the following sense: $\forall F, H \in \bar{X}$ and $0 < \alpha < 1$, if F is indifferent to H then both are indifferent to the c.d.f. $\alpha F + (1 - \alpha)H$. Now Assumption 4 was shown to imply that corresponding to each $(c_1', F) \in S_{c_1'}$ there is one and *only* one (c_1', F^*) pair, also in $S_{c_1'}$, lying on the *same* indifference curve. Repeating this process for other points, such as (c_1', G), on different indifference curves will be imagined to generate the diagonal TT'.

One can think of the family of linear indifference curves on $S_{c_1'}$ as being ordered by the one-point c.d.f. set $\bar{X}^*$ which can readily be seen to correspond to the natural order: $F^* > G^*$ iff the "saltus point" of F^* exceeds that of G^*. We shall adopt the *convention* that the value of the saltus point increases as one moves northeasterly along the diagonal TT'.

In order to simplify the discussion, let us assume that $\bar{X}$ is r.p.i. of C_1. This says that each of the "conditional" preorderings are identical, and thus in terms of Figure 1, the indifference curves in $S_{c_1''}$ will be parallel[13] to those in $S_{c_1'}$.

Consider the choice between (c_1', F) and (c_1'', G). As a first step, the conditional risk preference preorderings $\overset{S_{c_1'}}{\leqslant}$ and $\overset{S_{c_1''}}{\leqslant}$ can respectively be used to transform the choice into one between (c_1', F^*) and (c_1'', G^*). It is our contention that the resulting inter-cross-sectional choice is essentially a matter for "time" preferences with the question of risk having been "factored out".

But exactly what is meant by *"time" preferences* in the present context? It would seem consistent with the classic work of Irving Fisher [6] to associate this terminology with one's preferences over the set of certain two-period consumption plans, $\overset{C}{\leqslant}$. One can think of obtaining the DM's "time" preferences ($\overset{C}{\leqslant}$) by appropriately "subtracting" his "risk" preferences ($\overset{\bar{X}}{\leqslant}$) from $\overset{S}{\leqslant}$. The first step is to determine the set S^*, using just "risk" preferences. In terms of the *idealized* portrayal of Figure 2, S^* can be identified as the collection of diagonals, one for each cross section in S, which together form the diagonal plane $TT't't$ in the box S. One can then imagine trying to coordinatize this plane: measuring first-period consumption along Tt and one-point c.d.f.'s (c.d.f. saltus points) along TT'.

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[12] $\bar{X}$ is just some convex set and its representation geometrically as a rectangular cross-section should be viewed as nothing more than a convenient expository convention.

[13] If $\overset{S_{c_1'}}{\leqslant}$ and $\overset{S_{c_1''}}{\leqslant}$ are identical, TT' must be the same on both cross-sections and any $F \in \bar{X}$ will have the same indifferent one-point c.d.f. F^*. Thus the level curve containing F and F^* must intercept TT' in the same angle on $S_{c_1'}$ and $S_{c_1''}$.

Based on $\overset{S^*}{\lesssim}$, level curves could then be identified. But clearly this implies, via $(c_1, F^*) \mapsto (c_1, \hat{c}_2(F^*))$ (where $\hat{c}_2(F^*) \in C_2$), a "time preference" indifference map on $C_1 \times C_2$.

FIGURE 2

5. RELATING THE OCE AND TPC REPRESENTATIONS

Having presented our new representation of $\overset{s}{\lesssim}$ over the preceding sections, let us next compare it with the corresponding two-period cardinal approach. For this purpose we shall assume risk preference independence.[14] Now it is well-known that assuming $\overset{s}{\lesssim}$ to be "NM representable"[15] and $\underline{X}$ to be r.p.i. of C_1 implies that[16]

$$(4) \qquad \Psi(c_1, F) = EW(c_1, \tilde{c}_2^F) = \alpha(c_1) + \beta(c_1)EV(\tilde{c}_2^F), \quad \beta(c_1) > 0,$$

where Ψ and W are, respectively, the (two-argument) "Bernoulli" and "NM" indices. This particular form of W (or special cases thereof[17]) and hence the underlying "risk preference independence" postulate have been employed

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[14] Although it is straightforward to extend the results of this section to the more general case of risk preference dependence, doing so would needlessly complicate the basic ideas. Cf., Rossman and Selden [23, Section 5].

[15] Paralleling footnote 6, $\overset{s}{\lesssim}$ will be said to be "NM representable" iff there exists a continuous "Bernoulli" index Ψ, Ψ is "linear in the probabilities", and there exists a continuous TPC index W defined by

$$\Psi(c_1, F) = \int_{C_2} W(c_1, c_2)\, dF(c_2).$$

[16] See, for instance, Pollak [19], Raiffa [22], and Keeney [13].

[17] W will take the common additively separable form if $\beta(c_1) \equiv \beta$. A still more restrictive class of TPC indices is produced by assuming that $\alpha(c_1) = V(c_1)$, i.e., the DM's preferences are "stationary" (e.g., Hicks [10, 253ff.] or Fishburn [4, 96ff.]).

extensively in the management science and financial economics literatures.[18]

The following establishes the important result that the TPC is but a special case of the OCE representation of $\stackrel{s}{\lesssim}$:

THEOREM 2: *Given $\tilde{X}$ is r.p.i. of C_1, every TPC representation of $\stackrel{s}{\lesssim}$ can be transformed into an OCE representation, but the converse is not true.*

PROOF: To verify the first claim, note that if $\tilde{X}$ is r.p.i. of C_1, then equation (4) holds. But if for a given c_1, $\hat{c}_2$ is defined by $W(c_1, \hat{c}_2) = EW(c_1, \tilde{c}_2)$, then it easily follows that $EW(c_1, \tilde{c}_2) = \alpha(c_1) + \beta(c_1)V(\hat{c}_2)$. However, this clearly corresponds to an OCE representation defined by $U(c_1, c_2) = \alpha(c_1) + \beta(c_1)V(c_2)$, where V is identified as the period-two NM utility function.

To establish that the converse is false, note that $\stackrel{s}{\lesssim}$ being OCE representable (assuming r.p.i.) implies that

$$\mathcal{U}(F) =_{\text{def}} U(c_1, \hat{c}_2) = U(c_1, V^{-1} \int_{C_2} V(c_2)\, dF(c_2)).$$

The assertion that the OCE representation can be expressed as a TPC representation would imply that the former is "linear in the probabilities" which may be expressed as

$$(5) \qquad \mathcal{U}(\pi F + (1 - \pi)G) = \pi\mathcal{U}(F) + (1 - \pi)\mathcal{U}(G)$$

$\forall F, G \in \tilde{X}$ and $0 < \pi < 1$. To show that this need not be true consider the OCE representation defined by

$$U(c_1, c_2) = -1/c_1 - 1/c_2, \quad V(c_2) = -c_2^{-2}.$$

Simple computation will verify that for arbitrary $F, G \in \tilde{X}$ and $0 < \pi < 1$, equation (5) will not hold.   *Q.E.D.*

In comparison with assuming $\stackrel{s}{\lesssim}$ to be OCE representable, making the far more restrictive (in the sense of Theorem 2) assumption that this ordering can be represented by the two-attribute expected utility rule will next be shown to have *two very special consequences*. First, the TPC *subset* of OCE representations is characterized by each of its members exhibiting "linearity in the probabilities" (cf., the proof of Theorem 2).

The *second* consequence of assuming $\stackrel{s}{\lesssim}$ to be "NM representable" relates to a very fundamental behavioral implication. That additional axiomatic structure required to produce the *technical* simplification of "linearity in the probabilities" simultaneously *establishes an unplanned for interconnection between risk and time preferences.* Thus, in contrast to the OCE approach, one cannot

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[18] With respect to the former see, for instance, Keeney [11] and [12], and Raiffa [22], and with respect to the latter, Phelps [18], Levhari and Srinivasan [16], Hakansson [8], Samuelson [25], Pye [21], and Rubinstein [24].

prescribe risk and time preferences independently. The essential point is that this lack of freedom is *not* something which has been *explicitly modelled*, but rather is a kind of theoretic by-product.

To establish this claim, let us begin by noting that if one simultaneously assumes each $\overset{S_{c_1}}{\leqslant}$ to be NM representable (Assumption 4) and $\bar{X}$ to be r.p.i. of C_1, then the NM indices defined on the *set* of cross sections are related as follows:[19]

$$\{c_1\} \times \bar{X} : EV(\tilde{c}_2),$$

$$\{c_1'\} \times \underline{X} : \alpha' + \beta' EV(\tilde{c}_2), \quad \beta' > 0,$$

$$\{c_1''\} \times \bar{X} : \alpha'' + \beta'' EV(\tilde{c}_2), \quad \beta'' > 0,$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

which can be summarized by the following "*conditional risk preference function*":

(6) $\qquad \alpha(c_1) + \beta(c_1) EV(\tilde{c}_2), \quad \beta(c_1) > 0.$

Thus one's choice between any pair of points on a *single* cross-section such as (c_1', F) and (c_1', G) can be made according to whether

$$\alpha(c_1') + \beta(c_1') EV(\tilde{c}_2^F) \lessgtr \alpha(c_1') + \beta(c_1') EV(\tilde{c}_2^G).$$

Now this is true for *both* the OCE and TPC theories. Where they depart is in the *use of* (6) *for making inter-cross-sectional choices*. As should be clear from previous sections, one is totally unjustified under the OCE utility hypothesis in using the conditional risk preference function $_s$ to choose between points on different cross-sections. However, assuming $\overset{s}{\leqslant}$ to be "NM representable" implies that the "conditional risk preference function" *must* also be used for inter-cross-sectional choices as can be seen from comparing the relations (6) and (4). The function defined on (c_1, c_2)-pairs implicit in (6) then becomes the TPC von Neumann-Morgenstern index W, $W(c_1, c_2) = \alpha(c_1) + \underset{s}{\beta(c_1)} V(c_2)$. Now given that this $EW(c_1, \tilde{c}_2)$ is order-preserving with respect to $\overset{s}{\leqslant}$ and hence $\overset{s*}{\leqslant}$, W will represent the implied ordering over C. It is for this reason that remarks such as the following are often encountered:

REMARK 1 (Pollak [19]): An individual's ordinal utility function U and his von Neumann-Morgenstern index W, both defined on C, are closely related. Since they define the same indifference classes, each is an increasing monotonic transformation of the other.

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[19] I.e., if the preorderings on all of the cross-sections $(S_{c_1}, S_{c_1'}, S_{c_1''}, \ldots)$ are the same and "NM representable", then the corresponding NM indices can differ at most by a positive affine transform.

Thus, in a TPC framework, the representation of our DM's "time" preferences over certain consumption pairs must take the following form:

$$U(c_1, c_2) = T[\alpha(c_1) + \beta(c_1)V(c_2)], \quad T' > 0.$$

Hence as we set out to demonstrate, the DM's "time" preferences as reflected by U and his "risk" preferences as reflected by V cannot be prescribed independently. This is in contrast to the OCE approach.

In terms of *applying* the TPC and OCE utility theories to specific economic problems, the greater "freedom" of the latter in prescribing preference data could well prove important.

EXAMPLE: Suppose that an individual confronts a (risky) consumption/savings problem.[20] Let his "time" preferences be represented by

$$U(c_1, c_2) = (-c_1^{-\delta_1})/\delta_1 + (-c_2^{-\delta_1})/\delta_1, \quad -1 < \delta_1 < \infty,$$

and his (conditional) "risk" preferences by a period-two expected utility function with the (constant relative risk aversion) NM index

$$V(c_2) = -c_2^{-\delta_2}/\delta_2, \quad -1 < \delta_2 < \infty.$$

Then δ_1 and δ_2 are interpretable, respectively, as "time" and "risk" preference parameters. Under OCE preference theory δ_1 and δ_2 can be prescribed independently and their separate effects on optimal consumption and/or savings can be studied. In contrast, under the TPC theory δ_1 must equal δ_2 (cf., the proof of Theorem 2) and as a consequence the *separate* roles of time and risk preferences on optimal behavior cannot be distinguished.

This example serves to highlight one final point. As was noted in the discussion of Theorem 1, corresponding to each V, U-pair will be a $\overset{s}{\preceq}$ which is complete and a preordering. Thus by prescribing different (δ_1, δ_2)-pairs, and hence different V, U-pairs, one is free to construct any a priori sensible interrelationship between risk and time preferences.[21] But what is sensible? At least two criteria come to mind. First, does a given risk and time preference interdependence produce reasonable choices among feasible (c_1, F)-pairs? Second, in specific applications, such as the consumption/savings problem, is an economic agent's implied behavior (say first-period consumption) consistent with common sense and/or available empirical evidence?

6. FUTURE EXTENSIONS

As suggested earlier, this paper is but a first step in the study of the formation, from basic time and conditional risk preferences, of preferences over certain, uncertain consumption plans and the representation thereof. In order to apply

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[20] Cf., Selden [31] and Kihlstrom and Mirman [14].
[21] Included therein are the possible cases of independence and TPC dependence.

the OCE utility paradigm to a wide range of interesting problems, such as optimal economic growth and lifetime portfolio selection, it is first necessary to extend our preference theory to a multiperiod setting in which consumption is certain in the initial time period but random in each of T ensuing periods. Kreps and Porteus in [15] also question the appropriateness of the standard multivariate expected utility rule for problems of dynamic choice and propose an alternative axiomatization.

Another important extension is to the case of multiple goods. In this paper we have followed the more or less standard practice of assuming only a single (generalized) consumption good. However, it would seem highly desirable to generalize the basic OCE preference theory to a world with n commodities, and therewith investigate dynamic demand behavior with future period price uncertainty.

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REFERENCES

[1] ARROW, K. J.: *Essays in the Theory of Risk-Bearing.* Chicago: Markham Publishing Co., 1971.
[2] DEBREU, G.: *Theory of Value.* New Haven: Yale University Press, 1959.
[3] FAMA, E. F.: "Multiperiod Consumption-Investment Decisions," *American Economic Review,* 60 (1970), 163–174.
[4] FISHBURN, P. C.: *Utility Theory for Decision Making.* New York: John Wiley and Sons, 1970.
[5] ———: "Von Neumann-Morgenstern Utility Functions on Two Attributes," *Operations Research,* 22 (1974), 35–45.
[6] FISHER, I.: *The Theory of Interest.* New York: M. Kelly Publishers, 1930.
[7] GRANDMONT, JEAN-MICHEL: "Continuity Properties of a von Neumann-Morgenstern Utility," *Journal of Economic Theory,* 4 (1972), 45–57.
[8] HAKANSSON, N. H.: "Optimal Investment and Consumption Strategies Under Risk for a Class of Utility Functions," *Econometrica,* 38 (1970), 587–607.
[9] HARRIS, B.: *Theory of Probability.* Reading: Addison-Wesley, 1966.
[10] HICKS, J. R.: *Capital and Growth.* London: Oxford University Press, 1965.
[11] KEENEY, R. L.: "A Decision Analysis with Multiple Objectives; The Mexico City Airport," *The Bell Journal of Economics and Management Science,* 4 (1973), 101–117.
[12] ———: "An Illustrated Procedure for Assessing Multiattributed Utility Functions," *Sloan Management Review,* 14 (1972), 37–50.
[13] ———: "Utility Functions for Multi-attributed Consequences," *Management Science,* 18 (1972), 276–287.
[14] KIHLSTROM, R., AND L. MIRMAN: "Risk Aversion with Many Commodities," *Journal of Economic Theory,* 8 (1974), 361–388.
[15] KREPS, D. M., AND E. L. PORTEUS: "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica,* 46 (1978), 185–200.
[16] LEVHARI, D., AND T. N. SRINIVASAN: "Optimal Savings Under Uncertainty," *Review of Economic Studies,* 36 (1969), 153–163.
[17] PARTHASARATHY, K. R.: *Probability Measures on Metric Spaces,* London/New York: Academic Press, 1967.
[18] PHELPS, E. S.: "The Accumulation of Risky Capital: A Sequential Utility Analysis," *Econometrica,* 30 (1962), 729–743.
[19] POLLAK, R. A.: "Additive Von Neumann-Morgenstern Utility Functions," *Econometrica,* 35 (1967), 484–494.

[20] PRATT, J. W.: "Risk Aversion in the Small and in the Large," *Econometrica*, 32 (1964), 122–136.

[21] PYE, G.: "Lifetime Portfolio Selection with Age Dependent Risk Aversion," *Mathematical Methods in Investment and Finance*, ed. by G. P. Szego and K. Shell. Amsterdam: North-Holland Publishing Co., 1972, pp. 49–64.

[22] RAIFFA, HOWARD: "Preferences for Multi-Attributed Alternatives," Santa Monica: Rand Corporation, 1969.

[23] ROSSMAN, M., AND L. SELDEN: "The Existence of Two Attribute von Neumann-Morgen-stern Utility Functions Given Preferences on a 'Certain × Uncertain' Product Space," unpublished manuscript, University of Pennsylvania, 1976.

[24] RUBINSTEIN, M.: "The Strong Case for the Generalized Logarithmic Utility Model as the Premier Model of Financial Markets," *Journal of Finance*, 31 (1976), 551–571.

[25] SAMUELSON, PAUL A.: "Lifetime Portfolio Selection by Dynamic Stochastic Programming," *Review of Economics and Statistics*, 51 (1969), 239–246.

[26] SANDMO, AGNAR: "Capital Risk, Consumption and Portfolio Choice," *Econometrica*, 37 (1969), 586–599.

[27] SAVAGE, L. J.: *The Foundation of Statistics* (second revised edition). New York: Dover Publishing Inc., 1972.

[28] SCHUBERT, H.: *Topology*. Boston: Allymand and Bacon, Inc., 1968.

[29] SELDEN, L.: "A New Approach to the Joint Consumption/Portfolio Problem: Formulation, Graphical Analysis and Solution," Unpublished manuscript, University of Pennsylvania, 1976.

[30] ————: "A Note on the OCE Representation Hypothesis under Risk Preference Depen-dence," Unpublished manuscript, Columbia University, 1977.

[31] ————: "An OCE Analysis of the Effect of Uncertainty on Saving Under Risk Preference Independence," forthcoming in *Review of Economic Studies*.

[32] ————: "The Theory of Optimal Consumption and Portfolio Selection: A New Approach," unpublished Doctoral Dissertation, University of Pennsylvania, 1976.

[33] VON NEUMANN, J., AND O. MORGENSTERN: *The Theory of Games and Economic Behavior*. New York: John Wiley and Sons, Inc., 1967.