

Time Preferences, Conditional Risk Preferences, and Two-Period Cardinal Utility*

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1. INTRODUCTION AND SUMMARY

In recent years there has been extensive research on multiperiod decision problems in which economic agents are assumed to possess preferences over certain current/uncertain future consumption plans. Frequently, these problems are cast in a simplified two-period setting—see for instance, the papers of Drèze and Modigliani [2, 3], Leland [9], Sandmo [16-18] and Mirman [11], which deal with the consumption/savings (portfolio) problem. Almost without exception the agent's preferences are assumed to be representable by an expected TPC (Two-Period Cardinal) utility function, with the reader generally referred to the classic axiomatization of von Neumann and Morgenstern [22] (or Savage [19]).

The purpose of this paper is to investigate relationships among three types of preferences and their associated utility representations in a two-period context. The objects of choice are ordered pairs (x, F) in the product set $S \equiv C_1 \times \bar{X}$ where $C_1 = (0, \infty)$ is the set of certain consumption possibilities for the first period, and \bar{X} is the set of cumulative distribution functions on $C_2 = (0, \infty)$, the elements of which represent risky consumption

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possibilities for the second period. We consider the following three types of preference–utility structures:

(i) *time preferences* over certain consumption pairs in $C_1 \times C_2$ described by the binary relation \leq^t on $C_1 \times C_2$ and represented by a (continuous, strictly monotone) ordinal index $U: C_1 \times C_2 \rightarrow \mathbb{R}$ with $(c_1, c_2) \leq^t (c'_1, c'_2)$ iff $U(c_1, c_2) \leq U(c'_1, c'_2)$;

(ii) a set of *conditional risk preferences* $\{\leq_x \mid x \in C_1\}$, each of which is defined over \bar{X} and conditioned on a fixed level of first-period consumption and is representable according to the expected utility principle where $V_x: C_2 \rightarrow \mathbb{R}$ is a (continuous and strictly monotone increasing) second-period NM (von Neumann–Morgenstern) index such that, for all $F, G \in \bar{X}$,

$$F \leq_x G \text{ iff } \int_{C_2} V_x(c_2) dF(c_2) \leq \int_{C_2} V_x(c_2) dG(c_2) \quad (1.1)$$

(each V_x is unique up to a positive affine transformation); and

(iii) preferences over $S = C_1 \times \bar{X}$, described by the complete pre-ordering \leq and represented by a (continuous, strictly monotone) TPC (two-period cardinal) utility function $W: C_1 \times C_2 \rightarrow \mathbb{R}$ for which

$$(x, F) \leq (x', G) \text{ iff } \int_{C_2} W(x, c_2) dF(c_2) \leq \int_{C_2} W(x', c_2) dG(c_2). \quad (1.2)$$

The two-period NM index W is unique up to a positive affine transformation.

Assuming the existence of a two-period von Neumann–Morgenstern utility W , we consider in Section 2 when and how such a function can be constructed from a time preference index U and a *single conditional* NM utility function W_x . These results formalize a procedure suggested by Hirshleifer [7] and Drèze and Modigliani [3].

Unfortunately, even when the necessary conditions for constructing a TPC utility are satisfied, not every (perfectly standard) U and W_x pair is compatible mathematically and economically. Several interesting examples of incompatibilities are presented in Section 3. For instance, if time preferences are representable by a strictly quasi-concave ordinal utility function (from a subclass of the constant elasticity of substitution, or CES, family) and if the conditional index W_x exhibits risk aversion, then for x' infinitesimally larger than x the constructed conditional NM utility $W_{x'}(c_2)$ need not be risk averse everywhere and, in fact, can exhibit strong risk *affinity*.

Now instead of *assuming* the existence of a TPC utility function, one can ask whether it, in fact, is *implied* by the set of axioms sufficient for the existence of the ordinal time preference function U and a complete set of conditional second-period NM indices $\{V_x\}$. This is seen not to be the case. In Section 4, we identify that additional axiom, referred to as “coherence,” which is necessary and sufficient for the existence of a TPC utility function.

The coherence postulate essentially requires a very special kind of meshing together of the consumer's set of conditional risk preferences $\{\leq_x\}$ and time preferences \leq^t .

Our notion of coherence is formally similar to the concept of "time adjustment calculus" formulated by Prakash [13] for the case of "date-income" pairs. We comment on this paper briefly toward the end of Section 4. Some related issues also are discussed by Kreps and Porteus in their paper [8] on temporal resolution of uncertainty and by Fishburn [4] in the context of multilinear expected utility.

In the last section, the TPC utility model is compared with the OCE ("Ordinal Certainty Equivalent") representation hypothesis developed in Selden [20]. We show that the OCE representation of \leq is a two-period expected utility representation if and only if coherence is assumed. Since under the OCE theory given time preferences and conditional risk preferences are not incompatible (in the sense discussed above), it is the addition of coherence and hence the desire to have a TPC utility function which produces the possibility of incompatibilities.

2. CONSTRUCTION OF A TPC INDEX

Imagine a decision maker confronting a choice among alternative "certain x uncertain" consumption pairs. Suppose we *know* that a two-period expected utility function *exists*. Then how can it be constructed from the individual's "time" preferences and "risk" preferences?

To begin, we assume the following:

Assumption 1. There exists a complete preference preordering \leq on $C_1 \times \bar{X}$.

Assumption 2. The preordering \leq is NM representable as in Eq. (1.2), where W is the continuous TPC index.

It follows from (A.1) that there will exist a conditional preordering \leq_x on each subset $\{x\} \times \bar{X}$, where $x \in C_1$, and from (A.2) that each such ordering possesses an expected utility representation with $W_x: C_2 \rightarrow \mathbb{R}$ being the *conditional* NM utility. The collection of orderings $\{\leq_x\}$ is referred to as the individual's "risk preferences."

Turning next to "time preferences," a basic assumption required for the results in this section is

Assumption 3. There exists a complete preordering \leq^t on $C = C_1 \times C_2$ which is (i) continuous and strictly monotone and (ii) representable by the real-valued continuous time preference index $U: C \rightarrow \mathbb{R}$.

We make extensive use of the following notation to highlight an important characteristic of time preferences.

DEFINITION 1. If $x \in C_1$, then $C\{x\}$ denotes the class of pairs in C (temporally) indifferent to some pair in the set $C[x] =_{\text{def}} \{x\} \times C_2$; i.e.,

$$C\{x\} =_{\text{def}} \{(c_1, c_2) \in C \mid (c_1, c_2) \sim^t (x, c_2'), \text{ where } (x, c_2') \in C[x]\}.$$

As proved in Lemma 1 below, four types of regions may appear as $C\{x\}$ when \leq^t is continuous and monotone. This is illustrated in Fig. 1, where each type of region except the fourth is exhibited by a member of the CES class of utility functions: (1) $U(c_1, c_2) = c_1c_2$, (2) $U(c_1, c_2) = (c_1^{1/2} + c_2^{1/2})^2$, (3) $U(c_1, c_2) = (c_1^{-1} + c_2^{-1})^{-1}$. (For an interpretation of the elasticity of substitution as a measure of (intuitive) intertemporal complementarity cf., Selden [21].)

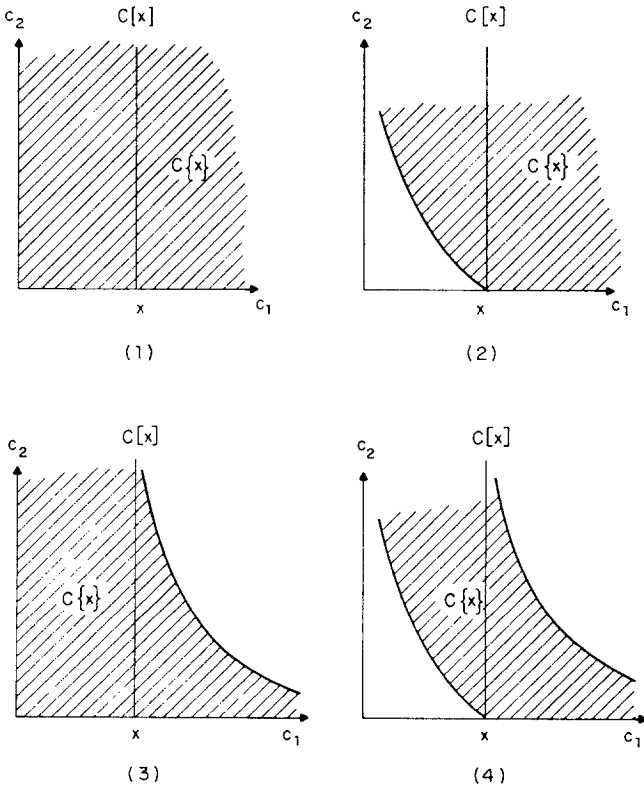


FIGURE 1

LEMMA 1. *If \preceq^t is continuous and strictly monotone, the set $C\{x\}$, $x \in C_1$, is one of the following four types:*

- (1) $C\{x\} = C_1 \times C_2$,
- (2) $C\{x\} = \{c \mid c' \prec^t c\}$ for some $c' \in C$,
- (3) $C\{x\} = \{c \mid c \prec^t c''\}$ for some $c'' \in C$,
- (4) $C\{x\} = \{c \mid c' \prec^t c \prec^t c''\}$ for some $c', c'' \in C$.

The proof follows from the continuity and strict monotonicity of \preceq^t and the connectedness of C .

Finally, let us explicitly relate the time preference and TPC utilities.

Assumption 4. The preference orderings \preceq^t and \preceq are related by the condition $W = h \circ U$, where h is some strictly monotone increasing function.

(In Section 4 we invoke an analogous assumption independent of the existence of W . Our present goal is to state a simple but quite useful relationship which must be satisfied by W , U , and a single arbitrary prespecified conditional NM utility W_x . For this purpose, Assumptions 1 through 4 are adequate.)

Given a conditional NM index W_x and a time preference relation on $C\{x\}$, the following formalizes a procedure suggested by way of an example in Hirshleifer [7, pp. 237-239]¹:

THEOREM 1. *Let $x \in C_1$ be arbitrarily chosen, and suppose (A.1)–(A.4) hold. Let W_x be the conditional NM utility function (corresponding to \preceq_x on $\{c_1\} \times \bar{X}$). Then for all $c \in C\{x\}$, W can be computed according to*

$$W(c_1, c_2) = W_x U_x^{-1}(U(c_1, c_2)), \quad (2.1)$$

where $x \in C_1$ and $U_x = U \mid C[x]$.

The proof is given in Appendix A. Thus the composition $W_x \circ U_x^{-1}$ defines the increasing monotonic transform h which distinguishes the TPC index from the time preference function U .

A number of brief comments about Theorem 1 are in order. First, it does *not* prove the existence of an expected utility representation; rather the theorem only provides a *computational formula* which is valid provided a TPC utility exists. Second, as can be seen from the proof, the restriction of the validity of the formula to the domain $C\{x\}$ is absolutely crucial. Third, whereas W_x and U_x^{-1} , in general, depend on x , h does not.

We shall use Theorem 1 to shed some light on interconnections posited by a TPC utility between \preceq^t and the collection of conditional risk preferences

¹ Hirshleifer credits Drèze and Modigliani [3] with introduction of this procedure. However, the latter seem more concerned with indirect utility for wealth than with *conditional* (period-two) expected utility for uncertain consumption.

$\{\leq_x \mid x \in C_1\}$. The first question for us to consider is whether a prespecified \leq^t and a single conditional risk preference relation \leq_x on $\{x\} \times \bar{X}$ may be *incompatible* with the *existence* of a two-period von Neumann–Morgenstern utility representation of \leq on *the whole* of $S = C_1 \times \bar{X}$.

3. INCOMPATIBLE TIME AND CONDITIONAL RISK PREFERENCES

We continue for the moment to suppose that the orderings \leq and \leq^t satisfy (A.1)–(A.4). Let $x \in C_1$ be preassigned and suppose that in C , $C\{x\}$ has an upper boundary (i.e., it is of type 3 or 4 in terms of Lemma 1). Any (ordinal) utility function U representing the corresponding time preferences must be *conditionally bounded* which, in turn, forces the boundedness of W_x .²

The impact of this boundary condition is illustrated by the following example, in which a specific (\leq^t, \leq_x) -pair is seen to be incompatible with the existence of a TPC representation of \leq .

EXAMPLE 1. Suppose time preferences are represented by $U(c_1, c_2) = (c_1^{-\delta} + c_2^{-\delta})^{-1/\delta}$, where $\delta > 0$. If $c_1 = x$, U_x is bounded by x . Assume further that for some x , the conditional risk preferences \leq_x are NM representable where the second-period NM index takes the form (up to a positive affine transform) $\log c_2$, which is *not* bounded above. According to (A.4) these utility functions must satisfy, for $c_1 = x$, the equation

$$\log c_2 = h(U(x, c_2)). \tag{3.1}$$

But since the log function is not bounded above and since U_x is bounded, the required relation (3.1) cannot hold for any increasing function defined on all $\text{im } U = (0, \infty)$. Therefore, NM representable preferences \leq over S cannot be compatible with the given \leq^t and \leq_x . This conclusion is echoed if we formally compute $W_x \circ U_x^{-1}(U) = \log(U^{-\delta} - x^{-\delta})^{-1/\delta}$, since the transform (1) is not defined for every possible value of U and (2) is not strictly increasing everywhere.³

The following result, complementary to Theorem 1, states simple compatibility requirements on \leq^t and a *single* \leq_x such that there can exist an ordering \leq on all of $S = C_1 \times \bar{X}$ satisfying (A.1)–(A.4).

² To see this, note first that $\sup U \mid C[x]$ is finite since there is a $c'' \in C$ such that for each $c \in C\{x\}$, $U(c) < U(c'')$. If \leq is NM representable, the conditional risk preference index W_x must also be bounded above. Assumption 4 guarantees that $W = h \circ U$ on C ; and, since $\sup U \mid C[x] < U(c'')$ for some $c'' \in C$, the monotone function h is bounded on $\text{im } U \mid C[x]$, forcing the boundedness of $W \mid C[x] = W_x$.

³ This example also illustrates the fact that the conditional NM index W_x may be bounded for reasons quite distinct from a desire to circumvent the St. Petersburg paradox (Arrow [1]).

THEOREM 2. Assume \leq^t satisfies (A.3), and let the initial condition $c_1 = x$ be given. Assume that the continuous, strictly monotone function $W_x: C_2 \rightarrow \mathbb{R}$ is a conditional NM utility for \leq_x over $\{x\} \times \bar{X}$. Then there exists on $S = C_1 \times \bar{X}$ an ordering \leq satisfying (A.1)–(A.4) compatible with \leq^t and \leq_x iff one of the following is true:

- (i) $C\{x\}$ is type 1;
- (ii) $C\{x\}$ is type 2, and W_x is bounded below;
- (iii) $C\{x\}$ is type 3, and W_x is bounded above; or
- (iv) $C\{x\}$ is type 4, and W_x is bounded below and above.

Only in case (i) must the \leq be unique.

The proof is given in Appendix B.

Remark. Though many preference relations on S are obtainable from a (U, W_x) -pair under cases (ii)–(iv) of Theorem 2, all of the resulting TPC, NM utilities coincide in $C\{x\}$.

Theorems 1 and 2 together provide the framework for investigating a second question. Given that \leq^t and \leq_x satisfy the conditions of Theorem 2 so that there exists a \leq for which (A.1)–(A.4) hold (i.e., \leq is “NM representable”), may they nevertheless be incompatible in some behavioral sense? For instance, will the resulting TPC index W exhibit undesirable properties?

We continue to assume CES time preferences

$$U(c_1, c_2) = (c_1^{-\delta} + c_2^{-\delta})^{-1/\delta}, \quad (3.2)$$

where $\delta > 0$ and the elasticity of substitution is given by $\eta = 1/(\delta + 1)$. We denote the two-period analogue of the Arrow–Pratt [1, 14] relative risk aversion function (in the spirit of Sandmo [16, 17]) by

$$\tau_R(c_1, c_2) =_{\text{def}} -c_2 W_{22}(c_1, c_2) / W_2(c_1, c_2), \quad (3.3)$$

and the conditional relative risk aversion function by

$$\tau_R^x(c_2) =_{\text{def}} -c_2 W_x''(c_2) / W_x'(c_2), \quad (3.4)$$

where $x \in C_1$. Suppose the conditional NM index assumes the form $W_x(c_2) = -\delta c_2^{-\beta} / \beta$, where $\beta > 0$ and $\tau_R^x(c_2) = \beta + 1$ (the constant value of relative risk aversion being denoted simply τ).⁴ Formal application of Theorem 1 yields an expression for W in $C\{x\}$ and hence one for $\tau_R(c_1, c_2)$ also in $C\{x\}$.

EXAMPLE 2. Let us assume the following: $\beta = 1$, $\delta = 2$, and $x = 10,000$.

⁴ Note that τ_R^x being constant does not imply that $\tau_R^{x'}$ ($x' \neq x$) will be.

Then the resulting TPC utility function may be described over $C\{x\}$ (but *not* over all of C) by $W(c_1, c_2) = -2(c_1^{-2} + c_2^{-2} - 10^{-8})^{1/2}$. For this function, direct computation reveals that $\tau_R^x = 2$ and $\eta = 1/3$. If we consider the point $c_1 = x = 10,000$, $c_2 = 22,400$ in $C\{x\}$, another direct computation using (3.3) yields $\tau_R(10,000, 22,400) = \tau_R^x(22,400) = 2$. However, if current consumption is increased by only 10% to $c_1 = 11,000$, then for $(11,000, 22,400) \in C\{x\}$ we obtain $\tau_R(11,000, 22,400) = -4.76!$ (See Fig. 2.)

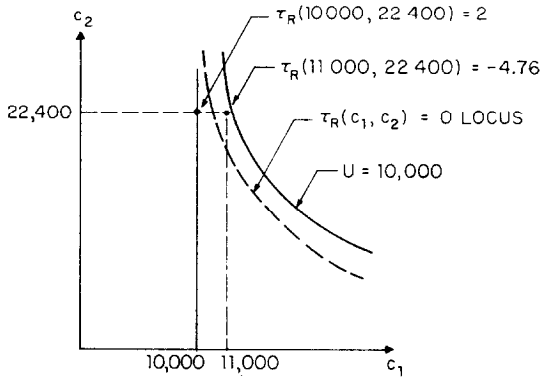


FIGURE 2

More generally, any values of the constant elasticity of substitution and relative risk aversion function satisfying $0 < \tau\eta < 1$ and $\eta < 1$ will produce preferences for which the property of being conditionally risk averse is unstable under small perturbations of first-period consumption. Yet this *seemingly pathological behavior of the TPC representation occurs when U and W_x exhibit perfectly "standard" properties.*

We leave to the interested reader consideration of other properties of the constructed TPC utility functions. Our only concern here is to indicate the possibility of behavioral incompatibilities between the time preferences and conditional risk preferences underlying two-period expected utility functions.

4. EXISTENCE

In this section we consider the existence of a TPC representation of preferences over $S(= C_1 \times \bar{X})$. Necessary and sufficient conditions are derived within a framework which directs attention to the interplay of conditional risk preferences and time preferences. By contrast, virtually all previous discussions of existence known to us proceed by applying directly the classical von Neumann–Morgenstern axiomatization. However, such an approach in the present setting confronts an immediate stumbling block in

providing the choice space S with a mixture structure. Specifically, if (c_1', F') is a mixture of (c_1, F) and (c_1', F') , how should c_1' be interpreted and/or defined when c_1 and c_1' are distinct? To form general “lotteries” appears to require dropping the distinction between current consumption being “certain” and future consumption being “risky” or “uncertain.” On the other hand, without a mixture structure on S , the NM axiomatization cannot be directly applied.

In the present discussion, we overcome this difficulty by relying solely upon mixtures defined only within the “slices” $S[x] = \{x\} \times \bar{X}$. Further, the classical NM axioms will be applied to *conditional* risk preferences, not to the relation \leq over S . Specifically, we relax Assumption 2 to read

Assumption 2'. For each $x \in C_1$, the conditional risk preference relation \leq_x (i.e., $\leq | S[x]$) is NM representable as in Eq. (1.1), with the continuous NM index V_x strictly monotonically increasing.

This assumption implies that we have a collection of second-period, conditional NM indices $\{V_x | x \in C_1\}$, but no longer have the TPC utility function W . As a result, it is necessary to use a reformulation of (A.4) which does not presume the existence of a W . To do so, first define the natural embedding $\iota: C \rightarrow S$ by mapping $(c_1, y) \mapsto (c_1, F_y^*)$ where F_y^* is the one-point c.d.f. with its saltus point at $y \in C_2$.

Assumption 4'. The preordering \leq satisfies the condition that time preferences are preserved under the natural embedding, i.e., $c \leq^t c' \Rightarrow \iota c \leq \iota c', \forall c, c' \in C$.

(It is easily verified that (A.1), (A.2), (A.3), (A.4) \Leftrightarrow (A.1), (A.2), (A.3), (A.4').)

For the ensuing existence discussion, we use the modified axiom structure (A.1), (A.2'), (A.3), and (A.4'). This represents a substantive weakening of the assumptions used in Sections 2 and 3, since the OCE preferences introduced in [20] and discussed in Section 5 satisfy this new axiom set but need not be NM representable.

The results of Sections 2 and 3 imply that the present modified set of axioms must be supplemented by some property interrelating conditional risk preferences. Theorem 1, in fact, suggests the nature of this interconnection. Let us rewrite Eq. (2.1) as

$$W_x(c_2) = W_{x'} U_{x'}^{-1}(U(x, c_2)) = W_{x'} U_{x'}^{-1} U_x(c_2),$$

where $x, x' \in C_1$. According to this expression, the conditional NM index W_x can be obtained from $W_{x'}$ by using $U_{x'}^{-1} U_x$. The function $U_{x'}^{-1} U_x$ provides a map from points in $C[x]$ to points in $C[x']$, which we call the *transfer* and denote γ .

DEFINITION 2. If $x, x' \in C_1$, the *transfer mapping* $\gamma: C[x] \cap C\{x'\} \rightarrow C[x']$ is characterized by the relation $c \sim^t \gamma c$, for each c in the domain of γ (cf., Fig. 3a). Let $S\{x\}$ denote the set of pairs (c_1, F) in S such that $\{c_1\} \times \text{supp } F \subset C\{x\}$, where $\text{supp } F$ is the support of F . The *induced transfer mapping* $S[x] \cap S\{x'\} \rightarrow S[x']$ associates to a pair (x, F) the pair (x', G) , where $F(y) = G(y')$ if $\gamma(x, y) = (x', y')$ (cf., Fig. 3b).

The induced transfer mapping will be denoted by the same symbol γ as that used for the transfer mapping.

If (x, F) is in the domain of the induced transfer γ , then the corresponding pair $(x', G) = \gamma(x, F)$ is obtained by "sliding along the intertemporal

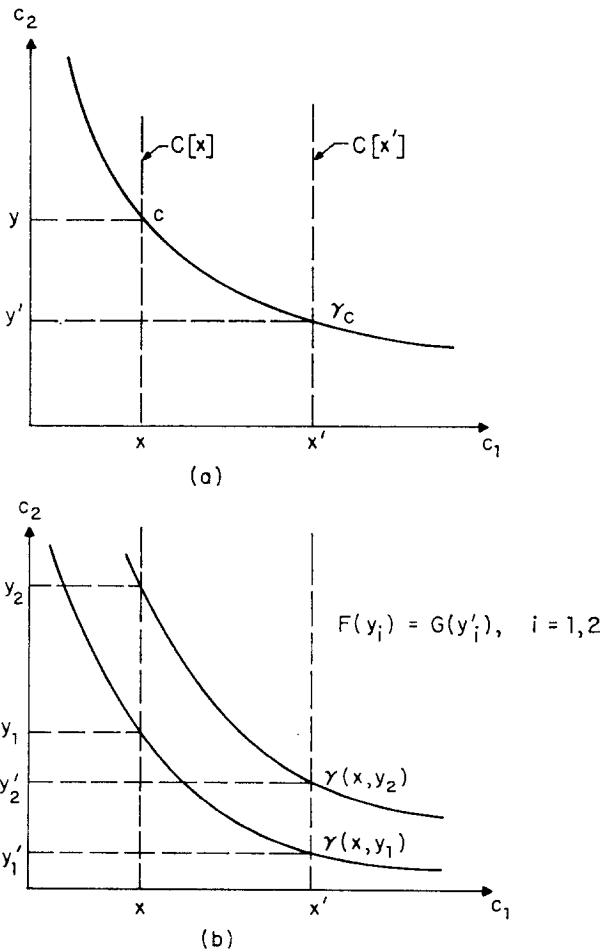


FIGURE 3

indifference curves." Thus for instance, suppose F has n -jump points $\{y_1, \dots, y_n\}$. Then under the induced transfer one constructs a c.d.f. G which (1) also has n -jump points, $\{y'_1, \dots, y'_n\}$, where each y'_i is obtained by finding that $(x', y'_i) \in C[x']$ lying on the same indifference curve as (x, y_i) and (2) has the same "probability structure" as F , i.e., $F(y_i) = G(y'_i)$ for each y_i and y'_i . Hence, the transfer is determined completely by the preference relation \leq^t . It is clear that the induced transfer is an affine mapping: $\gamma(x, aF + bG) = a\gamma(x, F) + b\gamma(x, G)$ where $a, b \geq 0$ and $a + b = 1$.⁵

The simplest interpretation of $W_x = W_x \cdot U_x^{-1} U_x$ involves asserting that the induced transfer maps $\leq | S[x]$ onto $\leq | S[x']$. More formally, we provide

DEFINITION 3. Conditional risk preferences $\{\leq_x | x \in C_1\}$ are *coherent* if $\forall s_1, s_2 \in S[x] \cap S[x'], s_1 \sim s_2 \Rightarrow \gamma s_1 \sim \gamma s_2$.

Assumption 5. Conditional risk preferences are coherent.

Coherence of the set of conditional risk preferences implies that the induced transfer γ maps a conditional indifference set in $S[x]$ into a conditional indifference set in $S[x']$. Since γ is determined purely by \leq^t , coherence is thus a property interrelating time and conditional risk preferences. As should be clear, this integrative property underlies both Theorem 1 and the existence of Examples 1 and 2 in Section 3. Moreover, it plays a crucial role in the proof of our basic two-period NM representation result.

THEOREM 3. (A.1), (A.2'), (A.3), (A.4'), and (A.5) are together necessary and sufficient conditions for the existence of a continuous, strictly monotone increasing TPC utility $W: C \rightarrow \mathbb{R}$.

The proof is to be found in Appendix C.

The relation $s \sim \gamma s$ which plays a crucial role in the proof of Theorem 3 can be given an interesting compensation interpretation. Consider a distribution F with probability mass concentrated at y_1 and y_2 so that $\text{Prob}\{y_1\} = p$, $\text{Prob}\{y_2\} = 1 - p$ (see Fig. 3b). Let $\gamma(x, F) = (x', G)$. We can regard $\gamma(x, F)$ as being obtained from (x, F) by a compensation process: using "time preferences" \leq^t , the increase $\Delta x = x' - x$ in current consumption is compensated by adjusting the levels y_1 and y_2 downward to y'_1 and y'_2 without altering the probability structure (i.e., without changing the value of p). So far we have assumed nothing which would imply that the conditional risk preferences on $S[x']$ are or are not compatible with γ . If the transfer completely compensates, according to the preference relation \leq , for the alteration

⁵ Note $(aF + bG)(y) = aF(y) + bG(y) = aF'(y') + bG'(y') = (aF' + bG')(y')$ by Definition 2.

Δx , then $(x, F) \sim \gamma(x, F)$. In this case, therefore, the utility of (x, F) determined by any index $S \rightarrow \mathbb{R}$ representing \leq would depend on the time preferences for the possible outcomes of the distribution, and the induced transfer mapping $\gamma: S[x] \cap S\{x'\} \rightarrow S[x']$ would map indifference sets into indifference sets.

Remark. Given the family $\{V_{c_1}\}$ of conditional NM indices, there is a naturally associated function V defined by $V(c_1, c_2) = V_{c_1}(c_2)$. This mapping $V: C \rightarrow \mathbb{R}$ must be interpreted with the greatest of care. The temptation is great to view this expression as a TPC utility. However, this would be *totally unwarranted* on the basis of (A.2'). As we have been arguing, (A.1), (A.2'), (A.3), and (A.4') are *not* sufficient to establish the existence of a TPC representation. One is only justified in using $EV(c_1, \tilde{c}_2)$ for choices between points in S characterized by a *common value of first-period consumption*. The fact that V depends on c_1 reflects a dependence of the second-period risk aversion (as, for instance, in the Arrow-Pratt sense) on the preceding period's level of consumption.

Remark. The thoughtful reader may well note the possibility of an alternative approach to the existence question. Let \mathcal{S} be the space of joint c.d.f.'s defined over $C_1 \times C_2$. Clearly, many elements of \mathcal{S} may not be meaningful economically (because whereas future consumption will in general be uncertain, current consumption will not be). Nevertheless \mathcal{S} is mathematically definable and supports a natural mixture structure which permits application of the traditional von Neumann-Morgenstern axioms (e.g., Fishburn [5]). The idea would then be to *embed* the economically meaningful world of \leq , S (where S is not a mixture space) into the mathematically "nice" world of \mathcal{S} and assert the following:

\leq is NM representable on S iff there is a complete preordering over \mathcal{S} which satisfies the traditional axioms and which agrees with \leq when restricted to S .

This result is easily verified. However, the approach may not be operationally meaningful since (i) the decision-maker may simply not possess preferences outside of S or (ii) he may have an ordering over all \mathcal{S} which, although NM representable over S , is not consistent with the NM axioms over other regions of \mathcal{S} . The embedding approach has still another disadvantage which is more significant economically: the traditional axioms for TPC utility are not known to reveal the interconnection between time and conditional risk preferences developed in this paper. Finally, the result provides no indication of a test which could determine when such an extension is possible; as a consequence, the condition given for existence would be difficult to verify or to disprove in any particular theoretical discussion.

Remark. The connection between time and NM preferences has also been investigated recently by Prakash [13], though in a quite different context. The notion of a “time adjustment calculus” is used to consider preferences over date-income pairs $(t, m) \in \mathbb{R}_+ \times \mathbb{R}$ where $\mathbb{R}_+ = [0, \infty)$. In addition to relatively minor differences in setting occasioned by Prakash’s use of the closed half-plane rather than the open positive orthant, more fundamental differences are produced by his use of the standard assumption that uncertainty is present at each time t . Nevertheless, his Theorem 3.5 (restricted to the domain of $\delta_{s,t}$) should be compared with our Theorem 1 (reformulated in terms of the transfer). Despite the differences in economic context, the results are mathematically analogous. However, Prakash’s argument depends upon a claim (his Proposition 3.4) which amounts to asserting that continuous, strictly monotone NM preferences over date-income pairs imply ordinal time preferences whose regions are only of type 1 (cf., Lemma 1 above). A counterexample in his context is provided by preferences defined by the utility $U(t, m) = (\exp(-2t) + \exp(-2m))^{-1}$, since the date-income pair $(\log 2, \log 2)$ is then superior to every income at time $t = 0$.⁶ Prakash examines neither the question of existence addressed by our Theorem 3 (possibly because in his setting mixtures are always available) nor that of the possible mathematical and/or behavioral incompatibilities considered in Section 3.

5. OCE UTILITY

In this section we show that the coherence postulate represents the conceptual link between TPC utility and the OCE (“Ordinal Certainty Equivalent”) representation of \preceq developed in Selden [20].

Let us first introduce some additional notation. Given a first-period consumption of x , the certainty equivalent period-two consumption associated with the c.d.f. F is denoted $\hat{c}_2(x, F)$. Then the “OCE Representation Theorem” can be stated as follows:

THEOREM 4 (Selden [20]).⁷ *Under (A.1), (A.2’), (A.3), and (A.4’), the ordering \preceq on S is OCE representable, in that $\forall x, x' \in C_1$ and $F, G \in \bar{X}$,*

$$(x, F) \preceq (x', G) \Leftrightarrow U(x, \hat{c}_2(x, F)) \leq U(x', \hat{c}_2(x', G)),$$

where $\hat{c}_2(x, F) = V_x^{-1} \int_{C_2} V_x(c_2) dF(c_2)$ and $\hat{c}_2(x', G) = V_{x'}^{-1} \int_{C_2} V_{x'}(c_2) dG(c_2)$.

⁶ The “discontinuity” argument in the proof of Proposition 3.4 incorrectly asserts that $(s, m') < l$ for all $p \in (0, 1]$.

⁷ Although the assumptions employed here are not exactly the same as those introduced in [20], the essential logic of the proof is.

In relating the OCE and TPC representations, we can state the following result as an immediate consequence of Theorems 3 and 4.

COROLLARY 1. *Under (A.1), (A.2'), (A.3), and (A.4'), any OCE representation can also be expressed as a TPC representation if and only if risk preferences are coherent.*

This result can be expressed in another, related way. If, in addition to (A.1), (A.2'), (A.3), and (A.4'), one invokes the coherence postulate, then corresponding to a given (U, V_x) -pair will be a two-period NM index W which differs from U by the transform $h = V_x \circ U_x^{-1}$. But *this* transform is exactly what is required to undo the nonlinear (in the probabilities) OCE representation:⁸

$$\begin{aligned} h \circ U \left(x, V_x^{-1} \int_{C_2} V_x(c_2) dF(c_2) \right) &= V_x \circ U_x^{-1} \circ U_x \left(V_x^{-1} \int_{C_2} V_x(c_2) dF(c_2) \right) \\ &= \int_{C_2} V_x(c_2) dF(c_2). \end{aligned}$$

We conclude this paper with a brief discussion of the implications of invoking the coherence axiom. Consider once again the case of the pairs (x, F) and (x', G) described in conjunction with Fig. 3b. One can compute the *conditional* certainty equivalent period-two consumption values as follows:

$$\begin{aligned} \hat{y} &= V_x^{-1}(\pi V_x(y_1) + (1 - \pi) V_x(y_2)), \\ \hat{y}' &= V_{x'}^{-1}(\pi V_{x'}(y'_1) + (1 - \pi) V_{x'}(y'_2)). \end{aligned}$$

On the basis of conditional risk preferences, (x, \hat{y}) will be indifferent to (x, F) , and (x', \hat{y}') to (x', G) . Now, in terms of Fig. 4 it is straightforward to see that if $(x, y_1) \sim^t (x', y'_1)$ and $(x, y_2) \sim^t (x', y'_2)$, then coherence *requires* that (x, \hat{y}) and (x', \hat{y}') lie on the same time preference indifference curve (and similarly for all such lotteries). In contrast, the OCE representation hypothesis would, in general, allow an indifference curve passing through (x, \hat{y}) to lie above or below (x', \hat{y}') . As this evidences, the coherence axiom in conjunction with the other assumptions produces a very strong interdependence between time and conditional risk preferences. It is as a consequence of this interdependence that the incompatibilities described in Section 3 arise. Under an OCE representation, a given $(U, \{V_x\})$ -pair will not exhibit such incompatibilities as long as coherence (which certainly possesses some intuitive appeal) is not assumed. (This, of course, is not to say that an

⁸ Clearly, no increasing monotonic transform of U , such as h , will affect the ordering \preceq (cf., [20, Corollary 1]).

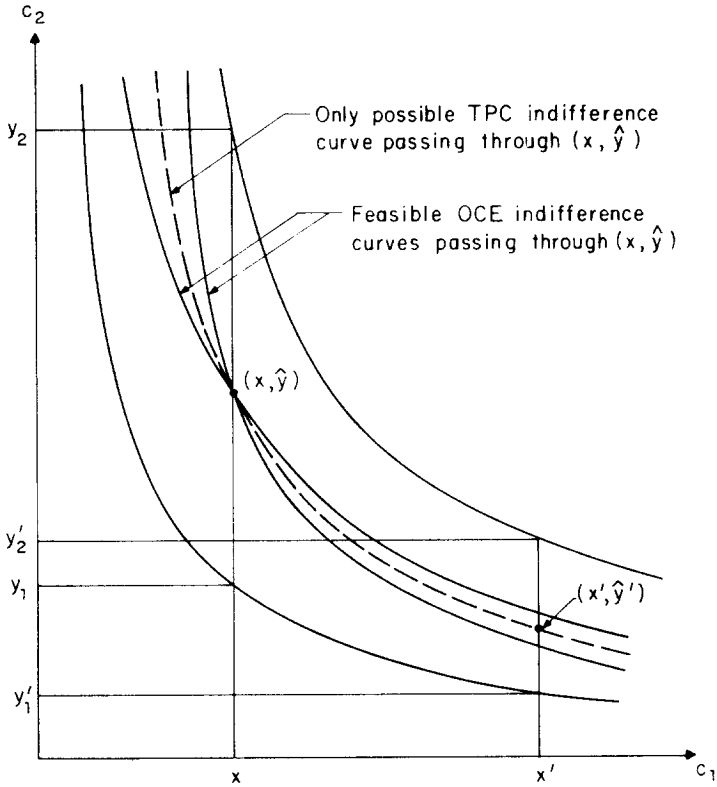


FIGURE 4

arbitrarily specified $(U, \{V_x\})$ -collection might not imply unreasonable behavior—for instance, in specific applications such as the consumption/savings problem.) As argued in [20], the OCE representation permits one to prescribe U and $\{V_x\}$ and hence model the interrelationship between time and risk preferences. In contrast, under the TPC approach, this interdependence is determined by a desire for the “mathematical simplification” of linearity in the probabilities, or equivalently by adoption of the coherence axiom.

APPENDIX

A. *Proof of Theorem 1.* Let T be a TPC index defined throughout C . Then by (A.4), we have $T = h \circ U$ for a strictly increasing function $h: \text{im } U \rightarrow \mathbb{R}$. By restricting consideration to $C\{x\}$ and noting that $\text{im } U \upharpoonright C\{x\} = \text{im } U_x$, we obtain $T \upharpoonright C\{x\} = h \upharpoonright \text{im } U_x \circ U \upharpoonright C\{x\}$. On the other hand, by restricting to $C[x]$, we obtain $T_x = h \upharpoonright \text{im } U_x \circ U_x$. Therefore, $h \upharpoonright \text{im } U_x =$

$T_x \circ U_x^{-1}$. Since T_x and W_x are both conditional NM utilities for \leq_x on $\{x\} \times \bar{X}$, we may write $T_x = A \circ W_x$, where A is affine. Consequently, $T|C\{x\} = A \circ W_x \circ U_x^{-1} \circ U|C\{x\}$. It follows that $W_x \circ U_x^{-1} \circ U|C\{x\}$ is also a TPC, NM index over $C\{x\}$. Q.E.D.

Observe that $h|im U_x$ is affinely equivalent to $W_x \circ U_x^{-1}$, which is not the same as having h affinely equivalent to $W_x \circ U_x^{-1}$ unless $im U_x = im U$. This distinction is at the root of all of our labor in proving Theorem 2 (Section 3), where we begin with a candidate for $h|im U_x$ and are required to obtain the extension h .

B. Proof of Theorem 2. The general line of argument establishing the necessity of one of the conditions (i)–(iv) is indicated in the discussion of Example 1. To establish sufficiency, i.e., existence of \leq , we shall consider the cases separately.

Case (i). Where $C\{x\} = C_1 \times C_2$, we have $im U_x = im U$. But if the monotone transform $h: im U_x \rightarrow \mathbb{R}$ is defined by $h = W_x \circ U_x^{-1}$, then $h \circ U$ is well defined. Consequently, the function $W = h \circ U$ defines a \leq on $S(=_{\text{def}} C_1 \times \bar{X})$ for which it is the TPC index. Moreover, the resulting W has W_x as its conditional NM utility for \leq_x (on $\{x\} \times \bar{X}$). The uniqueness assertion is clear from Theorem 1.

Case (iii). Again take $h = W_x \circ U_x^{-1}$. In this case the domain of h is equal to $im U_x$. But $im U_x$ is just the open interval $(inf U, sup U_x)$, where $sup U_x$ is finite and less than $sup U$. Moreover, since W_x is bounded above, h is also bounded above. Therefore, $\lim_{u \rightarrow sup U_x} h(u)$ exists. But this is exactly what is required to extend h to a monotone function $H: im U \rightarrow \mathbb{R}$. Now take \leq to be the relation over S having $W = H \circ U$ as its TPC index. This preordering is not unique because the extension H is not unique. (Note that if W_x and U are continuous, we can choose a continuous extension. Therefore, W is continuous if so desired.)

Cases (ii) and (iv) can be verified by a similar argument. Q.E.D.

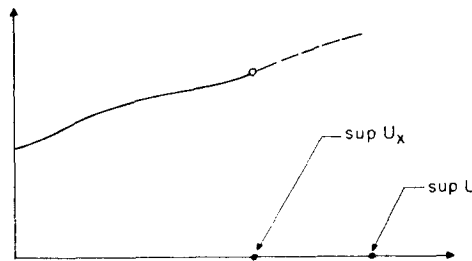


FIGURE 5

C. *Proof of Theorem 3.* The transfer mappings provide the essential structure which allows the existence result of classical NM theory to be extended to the certain first-period, uncertain second-period setting considered in this paper. At the level of riskless consumption pairs, the transfer mappings provide no more than a nonstandard, formal specification of \leq^t over $C = C_1 \times C_2$. The induced transfers form a class of extensions of the transfer concept to the level of conditional risk preferences: an extension characterized by the property that the induced mappings be *affine*. This structure can be taken into account provided it is compatible with the structure imposed by the a priori relation \leq on S . In particular, the transfer allows an affine representation on $S[x]$, where x is an arbitrary current consumption level, to be extended to $S\{x\}$ (which in some circumstances may be all of S). In the general case, the desired representation can be established throughout the entire choice space S by defining the representation over each set $S\{x\}$ of a collection of sets covering S , and by verifying that these definitions are identical where the covering sets $S\{x\}$ overlap.

Let us first define the following property of an induced transfer mapping.

DEFINITION. Risk preferences $\{\leq_s\}$ are *invariant* under transfer if $s \sim \gamma s$ for any s in $S[x] \cap S\{x'\}$.

In the context of this paper it is easy to show that coherence and invariance are logically equivalent properties.

The following lemmas show that the representation can be extended from $S\{x\}$ to the entire space S , and therefore that the TPC utility can be captured by restricting the extended affine index to the set $\iota(c_1, c_2)$ as in [6].

LEMMA 2. Given $x' \in C_1$, let λ be a representation of preferences on $S\{x'\}$ which is affine in the probabilities. Then a (slicewise) affine index $\tilde{\lambda}$ on $S\{x'\}$ is defined by $\tilde{\lambda}(s) = \lambda\gamma(s)$ for each $s \in S\{x\}$, where γ is the appropriate transfer.

Proof. Given that conditional risk preferences are coherent, as suggested above, they will also exhibit invariance, i.e., $\gamma s \sim s$ for each s . Therefore, $s_1 < s_2 \Rightarrow \tilde{\lambda}(s_1) < \tilde{\lambda}(s_2)$, and $s_1 \sim s_2 \Rightarrow \tilde{\lambda}(s_1) = \tilde{\lambda}(s_2)$. Therefore, $\tilde{\lambda}$ represents \leq on $S\{x'\}$. Since both λ and each transfer γ are affine, the extension $\tilde{\lambda}$ must be affine slicewise. \square

The above lemma establishes the desired representation for arbitrary regions of the form $S\{x\}$, $x \in C_1$. To extend this representation to all of S , a framework of subsets $C^{(N)}$, $N = 0, 1, 2, \dots$, is chosen such that (1) $C^{(N)} \subset C^{(N+1)} \subset C$, (2) $\bigcup_N C^{(N)} = C$, and (3) an affine representation over choices supported by $C^{(N)}$ can be extended to an affine representation over choices supported by $C^{(N+1)}$. Accordingly, the existence of an affine representation over all S is proved inductively. The following lemma provides the material for constructing $C^{(N)}$ as a union $\bigcup_{n=0}^N C\{x_n\}$ of fundamental equivalence

regions determined by the time preference relation \leq^t . The regions $C\{x_n\}$ are themselves inductively defined.

LEMMA 3. *There is a subset $\{x_n \mid n = 0, \pm 1, \pm 2, \dots\}$ of C_1 such that*

- (i) $C\{x_n\} \cap C\{x_{n+1}\} \neq \emptyset$ for each n ,
- (ii) $x_n < x_{n+1}$ for each n ,
- (iii) $\bigcup_n C\{x_n\} = C$.

Proof. Proceed inductively by choosing x_0 arbitrarily. It will suffice to demonstrate the extension of the chain $C\{x_{-N}\}, \dots, C\{x_N\}$ to the chain $C\{x_{-N-2}\}, \dots, C\{x_{N+2}\}$. If $C\{x_N\}$ has no right boundary, or if the right boundary approaches no point in $(0, \infty) \times \{0\}$, define x_{N+1} and x_{N+2} by $x_{N+1} = 1 + x_N$, $x_{N+2} = 2 + x_N$; if the right boundary approaches $(x, 0) \in (0, \infty) \times \{0\}$, then set $x_{N+2} = x$ and choose x_{N+1} to satisfy $x_N < x_{N+1} < x_{N+2}$. The choice of x_{-N-1} and x_{-N-2} is made in a similar way based on discrimination among left boundaries.

It is clear that the inductive step preserves properties (i) and (ii). Moreover, from the construction we shall prove that $x_n \rightarrow \infty$ as $n \rightarrow \infty$, and similarly $x_n \rightarrow 0$ as $n \rightarrow -\infty$. These limiting behaviors guarantee property (iii) of the claim.

Suppose $\sup x_n < \infty$. Then the left boundary of $C\{x\}$ must lie to the right of $C[x_n]$, where $x = \sup x_n$. Since these verticals approach $C[x]$, the left boundary must lie in $[x, \infty) \times C_2$. But strict monotonicity of \leq^t makes this impossible. Therefore $\sup x_n = \infty$.

In a parallel fashion, one may see that $\inf x_n = 0$. \square

Before proceeding to establish the principal inductive step of the extension proof, we need a preliminary result.

DEFINITION. If $K \subset C_1 \times C_2$, then $S(K)$ denotes the class of all $s \in S$ whose support lies in K .

With this notation, $S[x'] = S(\{x'\} \times C_2)$ and $S\{x'\}$ is $S(C\{x'\})$. Observe that $S(K \cap L) = S(K) \cap S(L)$ if $K, L \subset C$. Moreover, it is easy to show

LEMMA 4. *If $K, L \subset C_1 \times C_2$ are two nondisjoint open intervals determined by \leq^t , then each $s_1 \in S(K \cup L)$ is indifferent to an element $s_2 \in S(K) \cup S(L) \subset S(K \cup L)$.*

To prove the theorem, it is enough to establish

LEMMA 5. *For $N = 0, 1, 2, \dots$ there is a (slicewise) affine index λ_N representing \leq over $S(\bigcup_{-N}^N C\{x_n\})$ such that $\lambda_{N+1} \mid \text{domain } \lambda_N = \lambda_N$.*

Proof. Lemma 2 supplies λ_0 . Proceed inductively. Let $K = \bigcup_{-N}^N C\{x_n\}$,

$L = C\{x_{N+1}\}$. By Lemma 2 there is an affine index λ representing \leq over L . Since both L and K are intervals determined by \leq^t (use Lemma 1 and propositions (i) and (ii) of Lemma 3), $K \cap L$ is a subinterval of L . We propose to show that some affine modification of λ will coincide with λ_N in $S(K) \cap S(L)$ by using the classical NM uniqueness result. First view $S(K \cap L)$ as $S(\{x_{n+1}\} \times (y, \infty))$, where $y \in C_2$. Since $S(K \cap L) \subset S(L)$, both λ_N and λ provide affine indices representing \leq over $S(K \cap L)$; by classical expected utility theory, there is an affine transform τ such that $\tau \circ \lambda \upharpoonright S(K \cap L) = \lambda_N \upharpoonright S(K \cap L)$. Consequently, λ_{N+1} may be defined by

$$\lambda_{N+1}(s) = \begin{cases} \lambda_N(s), & s \in S(K), \\ \tau \circ \lambda(s), & s \in S(L). \end{cases}$$

According to Lemma 4, the extension to $S(K) \cup S(L)$ determines the extension to $S(K \cup L)$.

Two points must be verified: (1) λ_{N+1} represents \leq over $S(K \cup L)$, and (2) λ_{N+1} is slicewise affine.

To establish that λ_{N+1} represents \leq , consider s and s' . Suppose $s < s'$. Then we may suppose that $s = \iota c$, $s' = \iota c'$, where $c, c' \in K \cup L$. Since $s < s'$, it must be true that $c <^t c'$. Since we need not consider the cases where $\{c, c'\} \subset K$ or where $\{c, c'\} \subset L$, we may assume $c \in K$ and $c' \in L$. Now if $c' \notin K$, then $c \sim^t c''$ and $c' \sim^t c'''$, where $\{c'', c'''\} \subset C[x_{N+1}]$ and $c'' < c'''$. But the same may be said for any $c \in K$ such that $c \notin L$ and $c <^t c'$. Therefore, $\lambda_{N+1}(s) = \lambda_N(s)$, $\lambda_{N+1}(s') = \tau \circ \lambda(s')$, and $\tau \circ \lambda(s') \geq \sup \lambda_{N+1} > \lambda_{N+1}(s)$. Thus, $s < s'$ implies $\lambda_{N+1}(s) < \lambda_{N+1}(s')$. If $s \sim s'$, then $c \sim^t c'$. Thus $\{c, c'\} \subset K \cap L$, and clearly $\lambda_{N+1}(s) = \lambda_{N+1}(s')$. Now it follows immediately that $s \leq s'$ iff $\lambda_{N+1}(s) \leq \lambda_{N+1}(s')$.

Finally, to establish that an *affine* representation is possible, we use the classical uniqueness result once again to show that λ_{N+1} coincides with an affine transform over $S[x]$ for every x such that $S[x] \subset S(K \cup L)$. This clearly implies that the inductively defined representation over all of S is (slicewise) affine. First, note that there is some affine index φ over $S[x]$, for $x \in C_1$. But then φ and λ_{N+1} give affine indices over $S[x] \cap S(K)$. So $\lambda_{N+1} \upharpoonright S(K) \cap S[x] = \tau_1 \circ \varphi \upharpoonright S(K) \cap S[x]$. Moreover, $\lambda_{N+1} \upharpoonright S(L) \cap S[x] = \tau_2 \circ \varphi \upharpoonright S(L) \cap S[x]$. Consequently, $\tau_1 \circ \varphi \upharpoonright S(K \cap L) \cap S[x] = \tau_2 \circ \varphi \upharpoonright S(K \cap L) \cap S[x]$. Since $S(K \cap L) \cap S[x]$ contains at least two elements which are not indifferent, the affine transforms τ_1 and τ_2 must be identical. Therefore, $\lambda_{N+1} \upharpoonright [S(K) \cup S(L)] \cap S[x] = \tau \circ \varphi \upharpoonright [S(K) \cup S(L)] \cap S[x]$. Therefore, $\lambda_{N+1} \upharpoonright S[x] = \tau \circ \varphi$ when $S[x] \subset S(K \cup L)$. As φ is affine, λ_{N+1} must also be affine on each slice lying in $S(K \cup L)$. \square Q.E.D.

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