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ARROW–DEBREU PREFERENCES AND THE REOPENING OF CONTINGENT CLAIMS MARKETS

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The Arrow–Debreu intertemporal general equilibrium paradigm is typically interpreted as suggesting that contingent claims markets need not reopen as time passes and uncertainty resolves. We show that this property, if satisfied, has strong implications for the structure of agents' preferences and for the updating of probabilistic beliefs.

1. Introduction

This paper examines certain implications of the familiar Arrow–Debreu general equilibrium paradigm in a multiperiod uncertainty setting. In such models, agents' preferences are defined not only over goods today but also over goods deliverable in all future time periods for each possible state of nature. Markets for all such contingent commodities are assumed to exist at the current time. Under quite general conditions on preferences, an equilibrium is then established.

Our concern in this paper is with the widely held contention that if the Arrow–Debreu economy is characterized by a complete set of contingent claims markets, there will be no need for markets to reopen as time passes and uncertainty is resolved. We first show that this property of markets not reopening necessarily has two quite strong implications for

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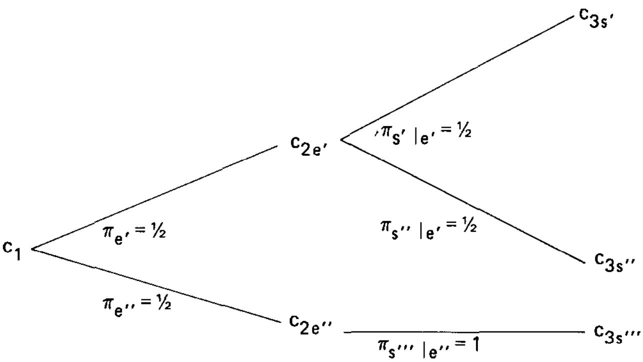

Fig. 1.

the participating agents' preference structures:

(i) agents' preferences in successive time periods will, in general, depend upon the (conditional) probabilities of events which, in the past, did not occur, and upon the probabilities of future states which, due to the resolution of uncertainty, can no longer occur; and
(ii) agents' preferences in successive time periods depend not only upon their actual consumption history, but also upon consumption plans for times and states which did not occur in the past, and which can no longer occur in the future.

Our work thus suggests that for general preference structures, the Arrow–Debreu paradigm implicitly assumes that agents choose to ignore information which will inevitably become available as time passes (such as the fact that the ex ante state probabilities should be revised over time to reflect the occurrence of certain events). We then show, in the context of a simple three period example, that if such information is, instead, used by agents in a natural way, there will be an incentive for markets to reopen at future times as uncertainty evolves. Hence, even in the presence of unchanging tastes, plans which are optimal in period one will not generally remain optimal in succeeding time periods. Finally, we show that this phenomenon of market reopening does not occur in the special case where all agents possess multiperiod von Neumann–Morgenstern preferences.

2. An example

As is standard in the Arrow–Debreu paradigm, let S denote the set of all mutually exclusive states of nature, where each state $s(=1,\ldots,|S|)$ corresponds to a complete specification of the history of the environment from the first time period to the last [see, for example, Guesnerie and de Montbrial (1974)]. This temporal structure leads to the notion of events: let E_t be the set of all period t events $e(=1,\ldots,|E_t|)$ which corresponds to a partitioning of S. With the passage of time, one discovers in which event $e \in E_t$ the true state s lies.

We consider two agents with *identical probabilistic* beliefs. Given their initial endowments, both individuals seek, through trading contingent commodities in period one, to allocate their wealth over three time periods. Let us further suppose a temporal stochastic setting characterized by the tree structure in fig. 1, where $S=\{s',s'',s'''\}$, $e'=\{s',s''\}$, $e''=\{s'''\}$, and $E_2=\{e',e''\}$, and where c_1 denotes period one consumption, $c_{2e'}$, $c_{2e''}$ denote, respectively, period two consumption contingent upon events e' and e'', $c_{3s'}$, $c_{3s''}$, $c_{3s'''}$ denote, respectively, period three consumption contingent upon states s', s'', and s''', and where $\pi_{e'}$, $\pi_{e''}$, $\pi_{s'|e'}$, $\pi_{s''|e'}$ and $\pi_{s'''|e''}$ denote the associated probabilities [see, once again, Guesnerie and de Montbrial (1974)]. Conditional on event e' or e'' occurring in period two, we wish to consider the possibility that the two agents may wish to trade again at that time.

Assume that the period one (ex ante) preferences of agent 1 are representable by the following Arrow–Debreu utility function:

$$U_1^1(c_1, c_{2e'}, c_{2e''}, c_{3s'}, c_{3s''}, c_{3s'''})$$

$$= -\left[\frac{1}{c_1}\right]^2 - \left[\frac{\pi_{e'}}{c_{2e'}} + \frac{\pi_{e''}}{c_{2e''}}\right]^2 - \left[\frac{\pi_{s'|e'}\pi_{e'}}{c_{3s'}} + \frac{\pi_{s''|e'}\pi_{e'}}{c_{3s''}} + \frac{\pi_{s'''|e''}\pi_{e''}}{c_{3s'''}}\right]^2$$

$$= -\left[\frac{1}{c_1}\right]^2 - \left[\frac{0.50}{c_{2e'}} + \frac{0.50}{c_{2e''}}\right]^2 - \left[\frac{0.25}{c_{3s'}} + \frac{0.25}{c_{3s''}} + \frac{0.50}{c_{3s'''}}\right]^2, \qquad (1)$$

where the superscript on U denotes the agent and the subscript the date at which the decision is made. Although (1) is clearly not a multiperiod expected utility function, it exhibits the quite standard properties of monotonicity, time additivity, conditional risk aversion and homotheticity. Next assume that agent 2's preferences are representable by the following expected utility function:

$$U_1^2(c_1, c_{2e'}, c_{2e''}, c_{3s'}, c_{3s''}, c_{3s'''})$$

$$= -\frac{1}{c_1} - \frac{\pi_{e'}}{c_{2e'}} - \frac{\pi_{e''}}{c_{2e''}} - \frac{\pi_{s'|e'}\pi_{e'}}{c_{3s'}} - \frac{\pi_{s''|e'}\pi_{e'}}{c_{3s''}} - \frac{\pi_{s'''|e''}\pi_{e''}}{c_{3s'''}}$$

$$= -\frac{1}{c_1} - \frac{0.50}{c_{2e'}} - \frac{0.50}{c_{2e''}} - \frac{0.25}{c_{3s'}} - \frac{0.25}{c_{3s''}} - \frac{0.50}{c_{3s'''}}. \tag{2}$$

In order to illustrate our basic point suppose that $(\hat{c}_1, \hat{c}_{2e'}, \hat{c}_{2e''}, \hat{c}_{3s'}, \hat{c}_{3s''}, \hat{c}_{3s'''})$ and $(\tilde{c}_1, \tilde{c}_{2e'}, \tilde{c}_{2e''}, \tilde{c}_{3s'}, \tilde{c}_{3s''}, \tilde{c}_{3s'''})$ denote, respectively, the optimal period one equilibrium allocations for agents 1 and 2, and further suppose that event e' occurs. The preferences of agents 1 and 2 for the remaining feasible consumption vectors $(c_{2e'}, c_{3s'}, c_{3s''})$ would then be described, under the Arrow–Debreu paradigm by, respectively:

$$U_{2e'}^1(c_{2e'}, c_{3s'}, c_{3s''}) = U_1^1(\hat{c}_1, c_{2e'}, \hat{c}_{2e''}, c_{3s'}, c_{3s''}, \hat{c}_{3s'''})$$

$$= -\left[\frac{1}{\hat{c}_1}\right]^2 - \left[\frac{0.50}{c_{2e'}} + \frac{0.50}{\hat{c}_{2e''}}\right]^2$$

$$-\left[\frac{0.25}{c_{3s'}} + \frac{0.25}{c_{3s''}} + \frac{0.25}{\hat{c}_{3s'''}}\right]^2, \tag{3}$$

$$U_{2e'}^2(c_{2e'}, c_{3s'}, c_{3s''}) = U_1^2(\tilde{c}_1, c_{2e'}, \tilde{c}_{2e''}, c_{3s'}, c_{3s''}, \tilde{c}_{3s'''})$$

$$= -\frac{1}{\tilde{c}_1} - \frac{0.50}{c_{2e'}} - \frac{0.50}{\tilde{c}_{2e''}} - \frac{0.25}{c_{3s'}} - \frac{0.25}{c_{3s''}} - \frac{0.50}{\tilde{c}_{3s'''}}. \tag{4}$$

It is clear that if markets were to reopen in the second period, conditional on e' having occurred, the equilibrium allocations would, once again, be given by $(\hat{c}_{2e'}, \hat{c}_{3s'}, \hat{c}_{3s''})$ and $(\tilde{c}_{2e'}, \tilde{c}_{3s'}, \tilde{c}_{3s''})$—hence it would have been unnecessary for the markets to have reopened. We see immediately, however, that the representation $U_{2e'}^1(\cdot)$ exhibits two strong properties: (i) it depends crucially upon elements of the original plan which have not been or cannot henceforth be realized $(c_{2e''}, c_{3s'''})$ (in the sense that the marginal rates of substitution among $c_{2e'}$, $c_{3s'}$, and $c_{3s''}$ are influenced by them), and (ii) it depends upon period one estimated probabilities of events which either have not occurred or can never occur ($\pi_{e''}, \pi_{s'''|e''}$).[1] Focusing on property (ii), it would seem more natural to suppose that agent 1 would, in period two conditional on e' having occurred, alter $\pi_{e'}$, and $\pi_{e''}$ from their ex ante values of $\frac{1}{2}$ and, $\frac{1}{2}$ respectively, to 1 and 0. Not to do so strikes us as somewhat arbitrary, since it effectively forces the agent to ignore information he knows he will possess when the period two decision must be made.

As an alternative to the Arrow–Debreu approach described above, we suppose that agents choose to incorporate in their period two preferences the information which they possess at time two conditional on e' having occurred; that is, that $\pi_{e''} = \pi_{s'''|e''}\pi_{e''} = 0, \pi_{e'} = 1, \pi_{s'|e'}\pi_{e'} = \pi_{s''|e'}\pi_{e'} = 0.5$. Given the same period one equilibrium allocations for agents 1 and 2 as before and conditional on e' having occurred, such adjustments yield the following second period preference structures (up to a monotonic transformation):

$$\overline{U}^1_{2e'}(c_{2e'}, c_{3s'}, c_{3s''} \,|\, \hat{c}_1, e')$$

$$= U^1_1\left(\cdot \,|\, c_1 = \hat{c}_1 \pi_{e''} = \pi_{s'''|e''}\pi_{e''} = 0;\ \pi_{e'} = 1;\ \pi_{s'|e'}\pi_{e'} = \pi_{s''|e'}\pi_{e'} = \tfrac{1}{2}\right)$$

$$= -\left[\frac{1}{c_{2e'}}\right]^2 - \left[\frac{0.50}{c_{3s'}} + \frac{0.50}{c_{3s''}}\right]^2, \tag{5}$$

$$\overline{U}^2_{2e'}(c_{2e'}, c_{3s'}, c_{3s''} \,|\, \tilde{c}_1, e')$$

$$= U^2_1\left(\cdot \,|\, c_1 = \tilde{c}_1;\ \pi_{e''} = \pi_{s'''|e''}\pi_{e''} = 0;\ \pi_{e'} = 1;\ \pi_{s'|e'}\pi_{e'} = \pi_{s''|e'}\pi_{e'} = \tfrac{1}{2}\right)$$

$$= -\frac{1}{c_{2e'}} - \frac{0.50}{c_{3s'}} - \frac{0.50}{c_{3s''}}. \tag{6}$$

Altering the state probabilities, as we have done, clearly involves no change in the agents' underlying preferences. Rather, it entails an explicit recognition of information which agents know they will possess in period two, given the realization e'. One argument for doing so is that the resulting period two representation $\overline{U}^i_{2e'}$ (or $\overline{U}^i_{2e''}$) obtained from U^i_1 is

[1] It follows from (4) that agent 2's period two preferences exhibit neither property (i) nor (ii). This is an immediate consequence of the state separability of his expected utility function (2).

independent of both the ex post irrelevant state probabilities and the ex post irrelevant consumption plans. [Another argument for our procedure can be developed along Bayesian lines so long as the period one preferences are *conditionally* von Neumann–Morgenstern – see Klein and Selden (1981).]

Incorporating the information concerning state probabilities which each agent possesses in period two has the effect, as we next show, of altering marginal rates of substitution between $c_{2e'}$ and $c_{3s''}$, assuming e' occurs. This in turn leads to a reopening of markets (a parallel argument holds if e'' occurs). Computing the two agents' period one ex ante marginal rates of substitution between $c_{2e'}$ and $c_{3s''}$ from eqs. (1) and (2) yields

$$\left.\frac{-\mathrm{d}c^2_{3s''}}{\mathrm{d}c^1_{2e'}}\right|_{U^1_1\text{const.}} = 4\left[\frac{1}{c^1_{2e'}}+\frac{1}{c^1_{2e''}}\right]\left[\frac{c^1_{3s''}}{c^1_{2e'}}\right]^2\Bigg/\left[\frac{1}{c^1_{3s'}}+\frac{1}{c^1_{3s''}}+\frac{2}{c^1_{3s'''}}\right], \tag{7}$$

and

$$\left.\frac{-\mathrm{d}c^2_{3s''}}{\mathrm{d}c^2_{2e'}}\right|_{U^2_1\text{const.}} = 2\left[\frac{c^2_{3s''}}{c^2_{2e'}}\right]^2, \tag{8}$$

where the superscripts distinguish the allocations of agents 1 and 2. Next we derive, from (5) and (6), the period two ex post (conditional on event e' having occurred) marginal rates of substitution for agents 1 and 2, respectively:

$$\left.\frac{-\mathrm{d}c^1_{3s''}}{\mathrm{d}c^1_{2e'}}\right|_{\bar U^2_{2e'}\text{const.}} = \frac{4\left[c^1_{3s''}\right]^2/\left[c^1_{2e'}\right]^3}{\left[\dfrac{1}{c_{3s'}}+\dfrac{1}{c_{3s''}}\right]}, \qquad \left.\frac{-\mathrm{d}c^2_{3s''}}{\mathrm{d}c^2_{2e'}}\right|_{\bar U^2_{2e'}\text{const.}}$$

$$= 2\left[\frac{c^2_{3s''}}{c^2_{2e'}}\right]^2. \tag{9}$$

Comparing (7), (8) and (9), we see that the ex ante and ex post $(c_{2e'}, c_{3s''})$ contract curves must, in general, differ because of the divergence in agent 1's period one and period two marginal rates of substitution (as a result of the ex ante dependence on the $c^1_{2e''}$ and $c^1_{3s'''}$ allocations). In contrast, were both agents to have von Neuman–Morgenstern preferences, it is clear that their marginal rates of substitution would not change and there would be no incentive to reopen markets.

Thus we have shown that if one of the agents possesses general Arrow–Debreu preferences [as in (1)] which are not von Neumann–Morgenstern, then the period one ex ante efficient allocation will not, in general, be ex post optimal in period two in the presence of probabilistic updating. As a result, there exists an incentive for markets to reopen in the second time period. Thus, the common presumption that a complete set of contingent commodity essentially reduces the dynamic uncertainty problem to a static, riskless one will, in general, only be true if one is willing to suppose additionally that agents' period two preferences depend on ex post irrelevant alternatives.

3. Concluding comments

As we have shown, agents' ex ante and ex post optimal allocations will not, in general, coincide because non-von Neumann–Morgenstern preferences (such as agent 1 in the previous section) will exhibit changing marginal rates of substitution as uncertainty is resolved over time. Hence agents will want to revise their ex ante optimal plans. But might not agents take this phenomenon into account in their strategic decision making? Provided agents recognize these facts, it seems reasonable to suppose that in forming their ex ante plans, they take into account their ex post choices they know ex ante they will make. Such a solution strategy is known as 'sophisticated' choice [following, for example, Pollak (1968)]. By construction it will be ex post optimal.[2] But the sophisticated allocation can also be thought of as being ex ante optimal *relative to the relevant information structure*, since for an agent not to follow a sophisticated strategy would suggest that he was ignoring freely available information.

Lastly, we briefly relate our results to those of Starr (1973). Provided agents possess von Neumann–Morgenstern preferences, Starr (1973) shows that agents' ex ante and ex post equilibrium allocations will agree if and only if they have identical probability beliefs. Our example shows that this result is not, in general, robust to arbitrary preference orderings.

[2] It is not difficult to show that although such a plan will not require revision, it is distinguishable from the standard Arrow–Debreu plans in not generally being rationalized by a transitive period one preference relation.

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