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ON THE RECOVERABILITY OF RISK AND TIME PREFERENCES FROM CONSUMPTION AND ASSET DEMANDS*

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We establish sufficient conditions for the recoverability and uniqueness of utility functions (preferences) generating consumption and asset demands in a two-period setting under uncertainty.

1. Introduction

The question of the relationship between a consumer's preferences and his observable behavior in competitive markets is one of the most thoroughly explored topics in modern economic theory [see, for instance, Chipman et al. (1971)]. One can investigate this relationship in two ways. First, demand relations can be derived by maximizing utility subject to appropriate budget constraints and then various properties of these demands implied by the optimization process can be studied. Alternatively, one can seek to 'rationalize' a given set of demand relations. This involves establishing conditions on the demand relations which imply the *existence* of a generating utility function (or more basically, preference ordering) and also specifying when there will be a *unique* generating utility and when it can be *recovered* from the known demands. Given that there exists a continuous, monotone, strictly convex preference relation (on the non-negative commodity space) which is 'lipschitzian', then following Mas-Colell (1977) these preferences will be representable (by a lipschitzian, weakly regular utility function), unique and recoverable.

Only very recently has the problem of 'rationalizing' consumer behavior in an uncertain setting been considered. Green, Lau and Polemarchakis (1979) establish sufficient conditions for a consumer's NM (von Neumann–Morgenstern) utility on end-of-period wealth to be *unique* within the class of func-

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tions which are analytic on $[0,\infty)$, and to be *recoverable* from knowledge of both his demand relations for m risky assets and the joint c.d.f. (cumulative distribution function) on gross asset returns. By introducing a risk-free asset, Dybvig and Polemarchakis (1981) are able to recover the consumer's NM utility without having to assume that it is analytic on $[0,\infty)$ and without having to assume full knowledge of the joint c.d.f. on asset returns (only the mean and variance for a single risky asset are required).

In this paper, we shall be concerned with the recoverability and uniqueness of utility functions (preferences) generating both asset demands and current consumption in a two-period setting. It is assumed throughout that the consumer possesses *time preferences* $\preccurlyeq^t$ defined on the space of certain consumption possibilities for periods one and two, $C \equiv C_1 \times C_2$ [where $C_t = (0,\infty)$ for $t = 1,2$], and represented by a (continuous and strictly monotone) ordinal index $U: C \to \mathbb{R}$. Also, for each $c_1 \in C_1$ there is defined on $\{c_1\} \times \mathfrak{X}$ (where $\mathfrak{X}$ is the set of c.d.f.'s on C_2) a *conditional risk preference* relation $\preccurlyeq_{c_1}$ which is representable according to the expected utility principle, with $V_{c_1}: C_2 \to \mathbb{R}$ denoting the (continuous and strictly monotone increasing) conditional, period-two NM index. Roughly speaking, these assumptions imply the existence of an OCE ('Ordinal Certainty Equivalent') preference relation $\preccurlyeq$ on the product space $S \equiv C_1 \times \mathfrak{X}$ [see Selden (1978) and section 2]. OCE preferences are especially helpful in making transparent the separate effects of risk and time preferences in intertemporal allocation problems such as the consumption-savings decision — see Selden (1979).

In order for the preference relation $\preccurlyeq$ to be NM representable not just conditionally but over the entire space S, additional axiomatic structure is required. It is necessary to assume that the conditional risk preferences defined by $\{V_{c_1} \mid c_1 \in C_1\}$ are 'coherent' in the sense that the following must hold for a given time preference index U and any pair of conditional NM utilities V_{c_1} and $V_{c_1'}$:

$$V_{c_1}(c_2) = V_{c_1'} U_{c_1'}^{-1} U_{c_1}(c_2). \tag{1}$$

[See Rossman and Selden (1978, sect. 4).] If this condition is satisfied, there will exist a continuous two-period NM utility $W: C \to \mathbb{R}$, which is unique up to a positive affine transform.

For the case of the joint consumption/portfolio allocation problem with both riskless and risky assets, we show that assuming OCE preferences and invoking essentially the same conditions on conditional NM preferences and gross asset returns as in Dybvig and Polemarchakis, one can uniquely recover *both* the underlying *time and risk preference indices* (Theorem 3). It follows as a corollary that if one imposes the stronger hypothesis that $\preccurlyeq$ is NM representable, then the two-period cardinal utility W can be uniquely identified.

Once we drop the assumption of a risk-free asset, it is necessary to impose stronger restrictions on conditional risk preferences and also to require additional information concerning the joint distribution of asset returns in order to recover the time and risk preference indices defining a generating OCE ordering (Theorem 4). Again, the case of $\preccurlyeq$ being NM representable follows as a corollary.

The conditions for recoverability in the consumption/savings decision problem differ significantly from those of the preceding cases due to the assumed existence of just a single risky asset. Whether $\preccurlyeq$ is OCE or NM representable, it is not possible in general to identify uniquely the generating utility functions from observed consumption or savings behavior (see Examples 2 and 3). However, in certain special cases recoverability can be restored. If $\preccurlyeq$ is OCE representable and the time preference index takes the following (ordinally) additively separable, discounted stationary form:

$$U(c_1, c_2) = u(c_1) + \delta u(c_2), \qquad \delta > 0,$$

then both the time and risk preference utilities can be uniquely identified (Theorem 5). Under the stronger hypothesis that $\preccurlyeq$ is NM representable, Theorem 6 establishes that U being just additively separable is sufficient for the two-period cardinal utility W to be recoverable. It should be noted that in a certain setting, Samuelson (1937) considers the related problem of recovering an individual's 'marginal utility of money income' from his income expenditures over time assuming a continuous time version of additive, discounted stationary utility.

Our results suggest the possibility of testing, from demand data, whether a given consumer's preference relation on $S \equiv C_1 \times \tilde{X}$ is NM representable.[1] Suppose that the necessary and sufficient conditions for $\preccurlyeq$ to be OCE representable hold and also that the time and risk preference indices are recoverable. Then in principle one could test whether these functions satisfy the coherence condition (1) and hence whether there exists an NM representation for $\preccurlyeq$. Now, of course, it would be preferable to replace any such 'in principle' test with derived restrictions on individual consumption and demand relations which are directly observable. The results obtained in this paper suggest the possible existence of such restrictions.

2. OCE preferences

In this section we summarize the key representation results for OCE and NM preference relations on $S \equiv C_1 \times \tilde{X}$. A more thorough exposition and proofs can be found in Selden (1978) and Rossman and Selden (1978).

[1]It is not our intention in this paper to present arguments for or against the coherence axiom, but rather to suggest the possibility of testing it.

Throughout this paper we take the perspective that the consumer's time preferences over certain consumption pairs, $\preccurlyeq^t$, and the collection of conditional risk preference relations, $\{\preccurlyeq_{c_1} | c_1 \in C_1\}$, constitute the basic preference data from which his ordering $\preccurlyeq$ over certain–uncertain consumption pairs in S is obtained. The conditional ordering $\preccurlyeq_{c_1}$ will be said to be *NM representable* if there exists a continuous 'NM index' $V_{c_1}: C_2 \to \mathbb{R}$ such that, for any period-two c.d.f.'s F, $G \in \mathfrak{X}$,

$$F \preccurlyeq_{c_1} G \Leftrightarrow \int_{C_2} V_{c_1}(c_2)\,\mathrm{d}F(c_2) \leqq \int_{C_2} V_{c_1}(c_2)\,\mathrm{d}G(c_2). \tag{2}$$

Assumption 1

(i) There exists a complete preordering $\preccurlyeq$ on $S \equiv C_1 \times \mathfrak{X}$.

(ii) The relation $\preccurlyeq$ induces a complete preordering $\preccurlyeq^t$ on $C \equiv C_1 \times C_2$,[2] which is continuous, strictly monotone and representable by the ordinal time preference index $U: C \to \mathbb{R}$.

(iii) For each $c_1 \in C_1$, $\preccurlyeq_{c_1}$ is NM representable with the continuous NM index V_{c_1} being strictly monotone increasing.

Under condition (ii) of Assumption 1, there will be a set of time preference indifference curves corresponding to U — see fig. 1. Condition (iii) states that on each 'cross section' $\{c_1\} \times \mathfrak{X}$ of S, the corresponding conditional ordering $\preccurlyeq_{c_1}$ is NM representable. This means that on each 'vertical' such as $\{c_1'\} \times C_2$ in fig. 1, there will be defined a conditional NM index $V_{c_1'}$. A given (c_1, F)-pair may then be thought of as a 'lottery' with c_2-payoffs along a single c_1-vertical. It is important to stress that being conditionally NM *does not imply* that the expected utility principle can be used for choices among (c_1, F)-pairs with different values of first-period consumption.[3]

Let us next introduce some additional notation. Given a first-period consumption of c_1, the certainty equivalent period-two consumption associated with the c.d.f. $F \in \mathfrak{X}$ is denoted $\hat{c}_2(c_1, F)$.

Theorem 1 [*Selden* (*1978*)]. *Under Assumption 1 the ordering* $\preccurlyeq$ *is OCE representable, in that* $\forall c_1, c_1' \in C_1$ *and* F, $G \in \mathfrak{X}$,

[2]Strictly speaking, we assume that $\mathfrak{X}$ is a mixture space and contains the set of one-point c.d.f.'s, denoted $\mathfrak{X}^*$, supported by the domain of $\mathfrak{X}$. Then it follows from condition (i) that there will exist a complete preordering defined over $C_1 \times \mathfrak{X}^* \subset S$. Let $F_y^*, G_z^* \in \mathfrak{X}^*$ be two *one*-point c.d.f.'s with their respective jump-points at y, $z \in C_2$. Then implicit in condition (ii) of Assumption 1 is the quite natural embedding property: $(c_1, F_y^*) \preccurlyeq (c_1', G_z^*) \Leftrightarrow (c_1, y) \preccurlyeq^t (c_1', z)$ for all $c_1, c_1' \in C_1$ and $F_y^*, G_z^* \in \mathfrak{X}$.

[3]Given the family $\{V_{c_1} | c_1 \in C_1\}$ of conditional NM indices, there is a naturally associated function $V: C_1 \times C_2 \to \mathbb{R}$ defined by $V(c_1, c_2) = V_{c_1}(c_2)$. As suggested in the text, one is not justified on the basis of Assumption 1 in using $\mathrm{E}V(c_1, c_2)$ for choices among points in S characterized by different c_1-values. The fact that V depends on c_1 simply reflects a dependence of conditional second-period risk preferences on the preceding period's level of consumption. See fig. 1.

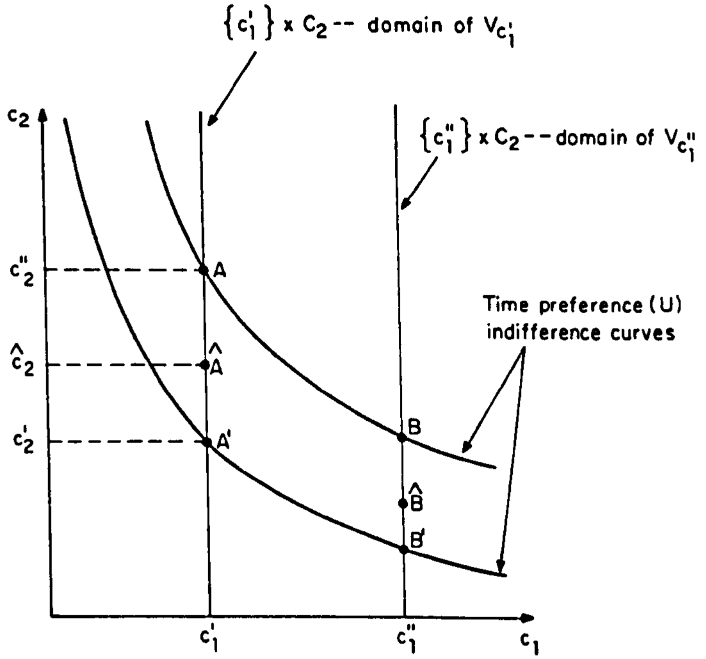

Fig. 1

$$(c_1, F) \preccurlyeq (c'_1, G) \Leftrightarrow U(c_1, \hat{c}_2(c_1, F)) \leqq U(c'_1, \hat{c}_2(c'_1, G)), \tag{3}$$

where $\hat{c}_2(c_1, F) = V_{c_1}^{-1} \int_{C_2} V_{c_1}(c_2)\, \mathrm{d}F(c_2)$ *and* $\hat{c}_2(c'_1, G) = V_{c'_1}^{-1} \int_{C_2} V_{c'_1}(c_2)\, \mathrm{d}G(c_2)$.

(Note that this representation is unique up to an increasing monotonic transform of U and a positive affine transform of V.) Thus if the three conditions of Assumption 1 hold, the ordering $\preccurlyeq$ will possess a continuous real-valued representation determined by the collection $(U, \{V_{c_1}\})$. The OCE utility of any (c_1, F)-pair is computed in two steps: first, using just the conditional NM index V_{c_1}, one converts (c_1, F) into a unique certain first-period, certainty equivalent second-period consumption pair, and then second one applies the ordinal time preference index to the resulting $(c_1, \hat{c}_2)$-pair.

It is readily apparent from (3) that OCE preferences will, in general, not be 'linear in the probabilities' as is required for $\preccurlyeq$ to be NM representable not just conditionally but over all of S. Given Assumption 1, the following is, however, necessary and sufficient for the existence of a continuous two-period NM utility $W: C \rightarrow \mathbb{R}$.

Assumption 2. Conditional risk preferences $\{\preccurlyeq_{c_1} | c_1 \in C_1\}$ are *coherent*; i.e., for a given time preference index U and an arbitrary pair of conditional NM utilities, eq. (1) holds.

[See Rossman and Selden (1978) for the appropriate 'region' restrictions on time preferences.]

Theorem 2 [*Rossman and Selden (1978)*]. *Let Assumption 1 hold. Then Assumption 2 is necessary and sufficient for* $\preccurlyeq$ *to be NM representable on S; i.e.,* $\forall c_1, c_1' \in C_1$ *and* $F, G \in \bar{X}$,

$$(c_1, F) \preccurlyeq (c_1', G) \Leftrightarrow \int_{C_2} W(c_1, c_2)\,\mathrm{d}F(c_2) \leqq \int_{C_2} W(c_1', c_2)\,\mathrm{d}G(c_2). \tag{4}$$

Thus in order for the OCE utility function to specialize to a two-period NM representation, the agent's time and conditional risk preferences must be coherent in the sense of eq. (1). This restriction can be interpreted quite simply in terms of fig. 1. Suppose that the certain consumption pairs A and B are indifferent under the time preference index U as are the pairs A' and B'. Consider a π-lottery of A and A' paying off c_2'' with probability π and c_2' with probability $(1-\pi)$ and an analogous π-lottery of B and B'. Let $\hat{A}$ and $\hat{B}$ denote the $(c_1, \hat{c}_2)$-pairs corresponding to the two π-lotteries. Then conditional risk preferences will be coherent if and only if $\hat{A}$ and $\hat{B}$ lie on the same time preference indifference curve. More generally, Rossman and Selden (1978, Theorems 1 and 3) show that this property is equivalent to eq. (1).

Both the general OCE and NM representations can be simplified considerably if the preference relation $\preccurlyeq$ exhibits the following *risk preference independence* (r.p.i.) property: $\forall c_1, c_1' \in C_1$ and $F, G \in \bar{X}$,

$$(c_1, F) \preccurlyeq (c_1, G) \Rightarrow (c_1', F) \preccurlyeq (c_1', G).$$

Assumption 3. The preference relation $\preccurlyeq$ exhibits r.p.i.

If this assumption holds, the OCE representation (3) simplifies to $U(c_1, V^{-1} \int V(c_2)\,\mathrm{d}F(c_2))$ and the two-period NM representation (4) simplifies to $\alpha(c_1) + \beta(c_1) \int V(c_2)\,\mathrm{d}F(c_2)$, where $\beta(c_1) > 0$.

Example 1. Let both Assumptions 1 and 3 hold and furthermore suppose that the time preference and second-period NM indices assume the following quite standard CES (constant elasticity of substitution) and constant relative aversion forms:

$$U(c_1, c_2) = (c_1^{-\delta} + c_2^{-\delta})^{-1/\delta} \quad \text{and} \quad V(c_2) = -c_2^{-\gamma}/\gamma,$$

where $-1 < \delta$, $\gamma < \infty$. Then for any $(c_1, F) \in S$, the OCE utility can be computed as follows:

$$U(c_1, V^{-1} \int V(c_2)\,\mathrm{d}F(c_2)) = [c_1^{-\delta} + (\mathrm{E}[c_2^{-\gamma}])^{\delta/\gamma}]^{-1/\delta}.$$

Now as shown in Selden (1979), this OCE representation will satisfy the coherence postulate and hence correspond to a two-period NM utility function if and only if $\gamma=\delta$, or equivalently, the Arrow–Pratt measure of relative risk aversion equals the reciprocal of the standard elasticity of substitution.

3. The consumption/portfolio problem

Consider an agent who must allocate his initial wealth among first-period consumption, c_1, and $m+1$ different assets or securities indexed by the subscript $j=0,1,\ldots,m$. Let x_j denote his holdings of asset j and r_j the random gross return on asset j. The c.d.f. of the random variable $r\equiv(r_0,\ldots,r_m)$ determines the c.d.f. of random second-period consumption $r\cdot x$ for any vector of asset holdings $x\equiv(x_0,\ldots,x_m)$ in $\mathbb{R}^{m+1}$.

Assumption 4. The random gross return variable r satisfies for any $j=1,\ldots,m$: (i) $\text{prob}\{r_j\geqq 0\}=1$, (ii) $\text{prob}\{r_j=0\}\neq 1$, (iii) r_j can not be written as a linear combination of $\{r_k\}$ for $k=0,1,\ldots,j-1,\ j+1,\ldots,m$[4] and (iv) $\text{E}r_j^l<\infty$, for $l=1,2$.

Assumption 5. The $j=0$ asset is riskless and has a gross return of 1; i.e., $\text{prob}\{r_0=1\}=1$.

Throughout this section we assume that, with respect to consumer preferences, Assumptions 1 and 3 hold. (The latter is introduced solely to simplify the analysis and can readily be dropped.) Thus, the consumer's preference relation over (c_1,F)-pairs is fully defined by the time preference and second-period NM index pair (U,V).

Assumption 6

(i) U is C^1 on $\mathbb{R}^2_{++}$, is quasiconcave and satisfies $\forall(c_1,c_2)\in\mathbb{R}^2_{++}$, $U_1(c_1,c_2)$, $U_2(c_1,c_2)>0$; and

(ii) V is C^2 on $\mathbb{R}_{++}$ and satisfies $\forall c_2\in\mathbb{R}_{++}, V'(c_2)>0$ and $V''(c_2)\leqq 0$.[5]

Without loss of generality, assume that there are only two assets, $j=0,1$, and that

$$\text{E}r_j=1,\qquad j=0,1. \tag{5}$$

[4]Strictly speaking, it suffices that two of the assets be linearly independent.

[5]Let y be some vector and let y^i denote a component of y. Then $y\geqq 0$ means $y^i\geqq 0$ for every i, $y>0$ means $y\geqq 0$ and $y\neq 0$ and $y\gg 0$ means $y^i>0$ for every i. $\mathbb{R}^n_+=\{y\in\mathbb{R}^n|y\geqq 0\}$ and $\mathbb{R}^n_{++}=\{y\in\mathbb{R}^n|y\gg 0\}$. We shall use $U_1(c_1,c_2)$ for $\partial U(c_1,c_2)/\partial c_1$ and $U_2(c_1,c_2)$ for $\partial U(c_1,c_2)/\partial c_2$. Also we write $V'(c_2)$ for $\text{d}V(c_2)/\text{d}c_2$ and $V''(c_2)$ for $\text{d}^2V(c_2)/\text{d}c_2^2$.

Faced with prices $(q,p)\equiv(q,p_0,p_1)\in\mathbb{R}^3_{++}$, the agent chooses $(c_1,x)\equiv(c_1,x_0,x_1)\in C_1\times X$ where $X=_{\text{def}}\{x|r\cdot x=c_2>0$ with probability one$\}$. The agent's initial wealth can be taken to be unity. Thus, given the time and risk preference indices (U,V), his problem can be expressed as

$$\max_{c_1,x} U(c_1,V^{-1}\mathrm{E}V(r\cdot x)),\quad \text{so that}$$
$$qc_1+p\cdot x=1,\qquad (c_1,x)\in C_1\times X. \tag{6}$$

Remark 1. Assumption 6 does not imply that the first-order conditions for the problem (6) are sufficient for a maximum. This can, however, be guaranteed in one of two ways: (a) by requiring $U(c_1,V^{-1}\mathrm{E}V(r\cdot x))$ to be quasiconcave in (c_1,x), or (b) by imposing on V and r the joint restriction in Selden (1980, Theorem 2).[6] We shall simply assume that the first-order conditions are sufficient for a solution to (6).

Remark 2. If one invokes the coherence Assumption 2, then it is easy to see that the agent's maximization problem specializes to the standard two-period expected utility formulation $\max_{c_1,x}\mathrm{E}[W(c_1,r\cdot x)]$; under risk preference independence it further simplifies to $\max_{c_1,x}\mathrm{E}[\alpha(c_1)+\beta(c_1)V(r\cdot x)]$.

At some prices (q,p) a solution to the maximization problem (6) may fail to exist.[7] Let $\mathscr{P}\subset\mathbb{R}^3_{++}$ be the set of prices for which a solution does exist. Define the consumption–asset demand correspondence $\xi:\mathscr{P}\to\mathbb{R}^3$ by $\xi(q,p)=(c_1(q,p),\ x_0(q,p),\ x_1(q,p))$. The correspondence ξ is observable and we assume knowledge of the distribution function for r.

We next show that the consumer's time preference index U and period-two NM utility V can both be recovered from ξ without ambiguity.

Theorem 3. Let both (U,V) and $(\tilde{U},\tilde{V})$ satisfy Assumption 6 and suppose that Assumptions 4 and 5 hold. If the same consumption-asset demand correspondence ξ is generated by (U,V) and $(\tilde{U},\tilde{V})$, then

$$U=T\circ\tilde{U},\qquad T'>0,$$

$$V=\alpha+\beta\tilde{V},\qquad \beta>0.$$

[6]As shown in Selden (1980), this restriction is satisfied, independently of the assumed return distribution, by the family of period-two NM indices resulting in 'portfolio separation'.

[7]The possibility that a solution to the maximization problem may fail to exist can be eliminated by extending the domains of definition of U and V to $\mathbb{R}^2_+$ and $\mathbb{R}_+$, respectively, and redefining C_t to equal $\mathbb{R}_+$, $t=1,2$. However we are interested in price systems for which the maximum is characterized by the interior first-order conditions. Consequently, the maxima added by 'closing from below' the consumption sets would later be ignored anyway.

Proof. The basic idea of the proof is to show that given our assumption it is possible to construct a $(\tilde{U}, \tilde{V})$-pair from ξ and moreover that any such pair of indices must be ordinally and cardinally equivalent, respectively, to the generating utility functions U and V.

By the concavity of the period-two NM utility V, $\mathrm{E}V(r\cdot x)$ is concave in x. The domain of definition for the latter, X, includes all x for which $r\cdot x = c_2$ is in the domain of definition of V with probability one. Assumptions 4(i), 5 and 6(ii) guarantee this whenever $x \in X^* \equiv \mathbb{R}_{++} \times \mathbb{R}_{+}$, i.e., whenever the agent invests a strictly positive amount of his initial wealth in the riskless asset. Furthermore, by the differentiability of V [Assumption 6(ii)] and the Lebesque Dominated Convergence Theorem, the derivative of $\mathrm{E}V(r\cdot x)$ with respect to $x_j, j=0,1$, exists and is given by

$$\partial\, \mathrm{E}V(r\cdot x)/\partial x_j = \mathrm{E}r_j V'(r\cdot x)$$

everywhere on X^* [see Green et al. (1979) for a detailed proof]. We can thus define the marginal rate of substitution of asset 0 for asset 1 by

$$S(x) = \mathrm{E}r_1 V'(r\cdot x)/\mathrm{E}V'(r\cdot x).$$

By the differentiability, monotonicity and concavity of V, and by the positivity and finiteness of the mean returns of the different assets, $S(x)$ is a strictly positive real number for all $x \in X^*$.

Notice now that, given $c_1^0 \in \mathbb{R}_{++}$ and $x \in X^*$, there exists a unique $(q,p) \in \mathbb{R}^3_{++}$ such that (c_1^0, x) solves the consumption/portfolio problem (6) at (q,p). The argument is as follows: By Assumption 6 the necessary first-order conditions for a maximum are

$$U_1(c_1, \hat{c}_2) = \lambda q, \qquad U_2(c_1, \hat{c}_2)\frac{\mathrm{E}r_j V'(r\cdot x)}{V'(\hat{c}_2)} = \lambda p_j, \qquad j=0,1, \tag{7}$$

where $\hat{c}_2 = V^{-1}\mathrm{E}V(r\cdot x)$. (Sufficiency is assumed — see Remark 1.) Existence follows by substituting (c_1^0, x) directly into the left-hand side of (7) and choosing λ such that $qc_1^0 + p\cdot x = 1$. To show uniqueness, we observe that U and V are differentiable and that c_1^0 and x_0 are strictly positive. As a consequence, the marginal rate of substitution, $S(x)$, is an observable function for $x \in X^*$. But then the derivatives of $S(x)$ must also be observable on X^*, provided they exist. Consider the expression

$$\frac{[\mathrm{E}V'(r\cdot x)\mathrm{E}r_1^2 V''(r\cdot x) - \mathrm{E}r_1 V'(r\cdot x)\mathrm{E}r_1 V''(r\cdot x)]}{[\mathrm{E}V'(r\cdot x)]^2},$$

which, if well-defined, would be the value of $\partial S(x)/\partial x_1$. At $\bar{a} \equiv (a, 0)$ in X^* we have $S(\bar{a})=1$ and

$$\frac{\partial S(\bar{a})}{\partial x_1}=\frac{V''(a)}{V'(a)}[\mathrm{E}r_1^2-(\mathrm{E}r_1)^2]=\frac{V''(a)}{V'(a)}\operatorname{var}(r_1). \tag{8}$$

For $\partial S(\bar{a})/\partial x_1$ to be well-defined, the right-hand side of eq. (8) must exist. But this follows from Assumptions 4(iv) and 5. Furthermore, by Hölder's inequality [see Royden (1968, p. 113)] $\mathrm{E}r_1^2 \neq (\mathrm{E}r_1)^2$ and hence the ratio $V''(a)/V'(a)$ can be determined. A simple argument, as in Pratt (1964), can then be employed to recover $\tilde{V}$ which will be affinely equivalent to the generating V.

Having recovered V up to a positive affine transformation, we next show that the ordinal representation of time preferences can be recovered up to an increasing monotonic transform. This is equivalent to recovering the marginal rate of substitution $U_1(c_1,c_2)/U_2(c_1,c_2)$ everywhere on $\mathbb{R}^2_{++}$.[8] As argued above, given any (c_1,x_0,x_1) in $C_1 \times X^*$ there exists a unique (q,p) such that (c_1,x_0,x_1) solves the agent's problem (6) at (q,p). Furthermore, this inverse demand function can be unambiguously derived from the correspondence ξ. Consider any $(c_1^0,c_2^0)\in C_1 \times C_2$. Setting $c_1=c_1^0$, $x_0=c_2^0$ and $x_1=0$, we see from the first-order conditions (7) that $U_1(c_1^0,c_2^0)/U_2(c_1^0,c_2^0)=q/p_0$ and hence is observable. Q.E.D.

Remark 3. Observe that complete knowledge of the joint c.d.f. for r is not necessary. It suffices to know just the mean and variance for a single risky asset.

Remark 4. If the consumption-asset demand correspondence ξ is known only for prices (q,p) on a subset of the domain $\mathbb{R}^3_{++}$, then, up to appropriate transforms, the generating (U,V)-pair can be recovered on a corresponding restricted subset of their domain of definition.

If the agent's ordering over certain–uncertain consumption pairs, $\preccurlyeq$, is NM representable, then it follows as an immediate corollary of Theorem 3 that the two-period NM index W can be uniquely recovered.

Corollary 1. Suppose that Assumptions 4 and 5 hold, and (U,V) and $(\tilde{U},\tilde{V})$ satisfy Assumptions 2 and 6. Then corresponding, respectively, to (U,V) and $(\tilde{U},\tilde{V})$ will be the two-period NM indices W and $\tilde{W}$. If the same correspondence ξ is generated by W and $\tilde{W}$, then

[8] That U can be uniquely recovered follows from Assumptions 1 and 6 and the argument in Mas-Colell (1977): Since U is quasiconcave, continuously differentiable and has strictly positive gradient everywhere on $\mathbb{R}^2_{++}$, it satisfies his conditions (Theorem 2′) for recoverability. Note that the continuous differentiability and strict positivity of the gradient imply that U is 'Lipschitzian' as defined by Mas-Colell (p. 1411).

$$W=\alpha+\beta\tilde{W}, \qquad \beta>0.$$

The preceding argument for the recoverability of the consumer's time and conditional risk preferences from his consumption–asset demand correspondence ξ hinges crucially on the assumed existence of a riskless asset. We next show that recoverability can be attained even in the absence of a riskless asset if we assume, alternatively, that the second-period NM utility V is analytic on $[0,\infty)$ — a rather strong assumption — and there are multiple risky assets each of which possesses finite moments of all positive orders. Again, without loss of generality, let $j=1,2$, and assume that $\mathrm{E}r_j=1$ for $j=1,2$.[9]

Assumption 7. For each $j=1,2$ and each positive integer l, $\mathrm{E}r_j^l<\infty$.

Assumption 8. The conditional period-two NM index V admits a unique extension which is analytic on $[0,\infty)$ and satisfies $V'(0)>0$.

Finally, in the absence of a riskless asset, the exclusion of the possibility of zero returns is natural, since preferences are defined only for strictly positive consumption vectors.

Theorem 4. Let both (U,V) and $(\tilde{U},\tilde{V})$ satisfy Assumptions 6 and 8 and suppose that Assumptions 4 and 7 hold and, furthermore, $\operatorname{prob}\{r_j=0\}=0$, $j=1,2$. *If the same correspondence ξ is generated by (U,V) and $(\tilde{U},\tilde{V})$, then*

$$U=T\circ\tilde{U}, \qquad T'>0,$$

$$V=\alpha+\beta\tilde{V}, \qquad \beta>0.$$

Proof. By essentially the same argument as in the proof of Theorem 3, $\mathrm{E}V(r\cdot x)$ is concave in x, is defined over the set X, and possesses the following derivatives

$$\partial \mathrm{E}V(r\cdot x)/\partial x_j=\mathrm{E}r_jV'(r\cdot x), \qquad j=1,2,$$

for all $x\in\mathbb{R}^2_+$. We can thus define the marginal rate of substitution of asset 2 for asset 1 by

$$S(x)=\mathrm{E}r_1V'(r\cdot x)/\mathrm{E}r_2V'(r\cdot x).$$

By the differentiability, monotonicity and concavity of V, and by the

[9] We use here $j=1,2$ instead of $j=0,1$ to index the two assets since, in the preceding argument, the index $j=0$ was associated with the riskless asset.

positivity and finiteness of the mean return of the different assets, $S(x)\in\mathbb{R}_{++}, \forall x\in\mathbb{R}^2_+\backslash\{0\}$.

Notice that given $c_1^0\in C_1$ and $x\in\mathbb{R}^2_+\backslash\{0\}$, there exists a unique $(q,p)\in\mathbb{R}^3_{++}$ such that (c_1^0,x) solves the maximization problem (6) at (q,p). By Assumption 6, the necessary first-order conditions for a maximum are

$$U_1(c_1,\hat{c}_2)=\lambda q, \qquad U_2(c_1,\hat{c}_2)\frac{\mathrm{E}r_j V'(r\cdot x)}{V'(\hat{c}_2)}=\lambda p_j, \qquad j=1,2, \tag{9}$$

where $\hat{c}_2=V^{-1}\mathrm{E}V(r\cdot x)$. (Sufficiency is assumed — see Remark 1.) Since V is strictly increasing and $x\in\mathbb{R}^2_+\backslash\{0\}$, there exists a unique $\hat{c}_2\in C_2$. By substituting (c_1^0,x) into the left-hand side of (9) and choosing λ such that $qc_1^0+p\cdot x=1$, we have existence. Uniqueness follows from the differentiability of U and V and the fact that c_1^0 and at least one of x_1 and x_2 is strictly positive. Thus, $S(x)$ is observable on $\mathbb{R}^2_+\backslash\{0\}$. But since it is well-defined and continuous on $\mathbb{R}^2_+$, it is observable throughout $\mathbb{R}^2_+$. Also, the derivatives of $S(x)$ of all orders at $x=0$ exist and are observable. This follows from the analyticity of V over $\mathbb{R}_+$, the finiteness of moments of all orders of both assets and Hölder's inequality.

Consider next the question of the uniqueness of the (U,V)-pair recovered from ξ within the class of all such pairs satisfying Assumption 8. Without loss of generality, we can suppose $V(0)=0$ and $V'(0)=1$. Rewrite the marginal rate of substitution expression as

$$S(x)\mathrm{E}r_2 V'(r\cdot x)=\mathrm{E}r_1 V'(r\cdot x). \tag{10}$$

Differentiating (10) with respect to x_1 and evaluating at $x=0$, we get

$$V''(0)[\mathrm{E}r_1r_2-\mathrm{E}r_1^2]=-\partial S(0)/\partial x_1. \tag{11}$$

The right-hand side of eq. (11) is observable and hence $V''(0)$ can be computed provided

$$[\mathrm{E}r_1r_2-\mathrm{E}r_1^2]\neq 0. \tag{12}$$

Similarly, reversing the roles of assets 1 and 2, we see that

$$[\mathrm{E}r_1r_2-\mathrm{E}r_2^2]\neq 0 \tag{13}$$

is also sufficient to compute $V''(0)$. But if both (12) and (13) fail,

$$(\mathrm{E}r_1r_2)^2=(\mathrm{E}r_1^2)(\mathrm{E}r_2^2),$$

which, by Hölder's inequality, can only occur if r_1 and r_2 are linearly dependent. But this contradicts Assumption 4. By repeated differentiation of eq. (10) and repeated application of Hölder's inequality, we can compute the higher order derivatives $V^{(k)}(0)$ for all $k \geqq 2$. Since V is assumed to be analytic on $\mathbb{R}_+$, the agent's second-period NM index has been recovered up to a positive affine transform.

Having recovered V, given any $(c_1, \hat{c}_2) \in \mathbb{R}^2_{++}$, we can find $x(\hat{c}_2) \in X$ such that $V(\hat{c}_2) = \mathrm{E}V(r \cdot x)$. Substituting c_1 and x in the first-order conditions (7) we see that

$$\frac{U_1(c_1, \hat{c}_2)}{U_2(c_1, \hat{c}_2)} = \frac{q}{p_j} \frac{V'(\hat{c}_2)}{\mathrm{E}r_j V'(r \cdot x)}, \qquad j = 1, 2,$$

and hence the ratio $U_1(c_1, \hat{c}_2)/U_2(c_1, \hat{c}_2)$ is observable everywhere on the domain of definition of U. Given the regularity Assumption 6, this is sufficient to recover the ordinal index U up to an increasing monotonic transform. Q.E.D.

Remark 5. As pointed out earlier, the r.p.i. Assumption 3 simplifies the notation and the argument, but is not, in any way, necessary for recoverability. The proofs of both Theorems 3 and 4 can be appropriately modified simply by changing each 'V' to 'V_{c_1}' — i.e., recovering first V_{c_1} for each $c_1 \in C_1$ and then U as before — and prefacing each argument with 'for all c_1'.

Remark 6. Observe that the argument for Theorem 4 requires knowledge of all moments of the return distribution for at least two assets.

Remark 7. Knowledge of the consumption–asset demand correspondence on an unbounded subset of the domain of prices may be required, since the point $x = 0$ may be attained only as a limit.

Remark 8. Now the case of $\preccurlyeq$ being NM representable follows as an immediate corollary of Theorem 4.

4. The consumption/savings problem

We next consider the question of recoverability for an agent who must allocate his initial wealth among first-period consumption and savings in a single risky asset. Let x denote his investment in the security and r its random gross return.

Assumption 9. The random gross return r satisfies: (i) $\mathrm{prob}\{r \geqq 0\} = 1$, (ii) $\mathrm{prob}\{r = 0\} = 0$, (iii) $\mathrm{prob}\{r = r^*\} \neq 1$, $\forall r^* \in \mathbb{R}_+$ and (iv) $\mathrm{E}r^l < \infty$ for each positive integer l.

Again throughout this section, Assumptions 1 and 3 are invoked; thus the consumer's preference ordering $\preccurlyeq$ is fully defined by the (U, V)-pair.

Assumption 10.

(i) U is analytic on $\mathbb{R}^2_{++}$, is quasiconcave and satisfies $\forall (c_1, c_2) \in \mathbb{R}^2_{++}$, $U_1(c_1, c_2)$, $U_2(c_1, c_2) > 0$;

(ii) V is analytic on $\mathbb{R}_{++}$ and satisfies $\forall c_2 \in \mathbb{R}_{++}$, $V'(c_2) > 0$ and $V''(c_2) \leqq 0$;

(iii) both U and V admit unique extensions to the closure of their domain of definition (i.e., $\mathbb{R}^2_+$ and $\mathbb{R}_+$, respectively) which are analytic and satisfy $U_1(0,0) > 0$, $U_2(0,0) > 0$ and $V'(0) > 0$.

Faced with prices $(q, p) \in \mathbb{R}^2_{++}$, the agent chooses $(c_1, x) \in C_1 \times X$ where $X =_{\text{def}} \{x \mid rx = c_2 > 0$ with probability one$\}$. The agent's initial wealth and Er can both be assumed to equal unity. Then his consumption/savings problem can be expressed as

$$\max_{c_1, x} U(c_1, V^{-1}\text{E}V(rx)), \quad \text{so that} \quad qc_1 + px = 1, \qquad (c_1, x) \in C_1 \times X. \tag{14}$$

Assumptions 9 and 10 are the exact analogues of Assumptions 4 and 6–8 employed in the previous section to yield recoverability in the absence of a riskless asset. In the present context of just a single asset, however, the previous argument fails. To gain some intuition for why this happens, note first that in the presence of two or more assets, it is possible to use the observation of the agent's allocation of wealth among the various assets for a fixed level of period-one consumption to reveal his second-period NM utility V. Once this index has been recovered, the certainty equivalent for any portfolio can be computed, and then the ordinal time preference index can be recovered by observing (indirectly) the agent's choices between $(c_1, \hat{c}_2)$-pairs. But now it is impossible to carry out the analogous argument in the case of just a single asset. For as soon as the first-period consumption level is fixed, the investment decision disappears — there are no alternative asset mixes to choose among.

The following example demonstrates that Assumptions 9 and 10 alone are not sufficient to yield the unique recoverability of risk and time preferences.

Example 2. Consider an agent with OCE preferences defined by the following ordinal utility and second-period NM index:

$$U(c_1, c_2) = f(c_1) + \alpha(\log[(c_2 + 1)^\beta - (1 - \gamma)] - \log \gamma),$$

$$V(c_2) = (c_2 + 1)^\beta,$$

where $0<\beta<1$, $0<\alpha$, $0<\gamma<1$ and $f(c_1)$ is analytic and concave on $\mathbb{R}_+$. (Note that U is analytic on $\mathbb{R}^2_+$.) Further suppose that the random return on the risky asset is distributed according to

$$\begin{aligned} r&=R \quad \text{with probability} \quad \pi, \\ &=0 \quad \text{with probability} \quad 1-\pi. \end{aligned} \tag{15}$$

Finally, let $\gamma=\pi$. Then the agent's objective function for the consumption/savings problem (14) becomes

$$U(c_1, V^{-1}\mathrm{E}V(rx))=f(c_1)+\alpha\beta\log(Rx+1).$$

But this implies that the consumption–asset demand correspondence depends only on the product $\alpha\beta$. Consequently, it is not possible to recover the agent's (U, V)-pair.

Observe that the ordinal intertemporal utility function considered in the example is of the *additive* form

$$U(c_1,c_2)=u_1(c_1)+u_2(c_2). \tag{16}$$

Consequently, assuming that the agent's time preferences possess an additive representation — in addition to Assumptions 1, 3, 9 and 10 — is not sufficient to guarantee recoverability. However, if besides being (ordinally) additively separable, U takes the discounted stationary form, i.e.,

$$U(c_1,c_2)=u(c_1)+\delta u(c_2), \qquad \delta>0, \tag{17}$$

the agent's risk and time preferences can be recovered from the consumption–asset demand correspondence $\xi(q,p)=(c_1(q,p),x(q,p))$.

Theorem 5. Suppose Assumptions 9 and 10 hold. Further, let the representation of time preferences take the (ordinally) additively separable, discounted stationary form (17). If the same consumption–asset demand correspondence ξ is generated by (u,δ,V) and $(\tilde{u},\tilde{\delta},\tilde{V})$, then

$$\delta=\tilde{\delta},$$

$$u=a+b\tilde{u}, \qquad b>0,$$

$$V=\alpha+\beta\tilde{V}, \qquad \beta>0.$$

Proof. We need only outline the argument, since it parallels the proof of

Theorem 4. The necessary first-order conditions for a maximum are

$$u'(c_1)=\lambda q, \qquad \frac{\delta u'(\hat{c}_2)\mathrm{E}rV'(rx)}{V'(\hat{c}_2)}=\lambda p,$$

where $\hat{c}_2=V^{-1}\mathrm{E}V(rx)$. (Again, sufficiency is assumed.) The marginal rate of substitution between first-period consumption and the risky asset is given by

$$S(c_1,x)=\frac{\delta u'(\hat{c}_2)[V'(\hat{c}_2)]^{-1}[\mathrm{E}rV'(rx)]}{u'(c_1)}. \tag{18}$$

It follows from Assumptions 9 and 10 that $S(c_1,x)$ is well-defined, observable and analytic on $\mathbb{R}^2_+$. Without loss of generality, we may set $u'(0)=1$, $V(0)=0$ and $V'(0)=1$. Then $S(0,0)=\delta$, and hence the discount factor, δ, can be recovered.

Now rewrite eq. (18) as follows:

$$u'(c_1)S(c_1,x)=\delta u'(\hat{c}_2)[V'(\hat{c}_2)]^{-1}[\mathrm{E}rV'(rx)]. \tag{19}$$

Differentiating both sides with respect to c_1 and then evaluating at $c_1=\hat{c}_2=x=0$, we get

$$\delta u''(0)=-\partial S(c_1,x)/\partial c_1,$$

and hence $u''(0)$ can be recovered. By repeated differentiation, we can recover the derivatives of u of all orders at $c_1=0$. Since u is assumed to be analytic on $\mathbb{R}_+$ (i.e., we assume the existence of a transform of U which is both additive and analytic), this is equivalent to recovering u up to a positive affine transform.

It remains to recover V. This follows by differentiating both sides of eq. (19) with respect to x and then paralleling the argument in the proof of Theorem 4. Q.E.D.

In order to recover the (U,V)-pair, we made the very strong assumption that the representation of time preferences takes the additively separable and discounted stationary form (17). It seems natural to ask whether by imposing the additional restriction that the consumer's conditional risk preferences are coherent (or equivalently that $\preccurlyeq$ is NM representable), one can dispense with the additive separability and/or stationarity of U and still have recoverability. The following example demonstrates that the assumption of additive separability can *not* be dropped as a requirement for recoverability.

Example 3. Let the consumer possess NM preferences over S defined by the two-period NM index

$$W(c_1,c_2)=\alpha f(c_1)+f(c_1)V(c_2),$$

where $\alpha\in\mathbb{R}$, $0<[\alpha+V(0)]/V(0)<1$, f and V are non-negative and analytic on $\mathbb{R}_+$ and W is monotone and concave. As in the previous example, let the random return on the risky asset be distributed according to (15). Further, suppose $\pi=[\alpha+V(0)]/V(0)$. Then the agent's objective function for the consumption/savings problem (14) becomes

$$\mathrm{E}W(c_1,rx)=\pi f(c_1)V(Rx). \tag{20}$$

But now consider a different two-period NM utility function

$$\tilde{W}(c_1,c_2)=\tilde{\alpha}[f(c_1)]^\gamma+[f(c_1)]^\gamma[V(c_2)]^\gamma,$$

where $\tilde{\alpha}$, $\gamma\in\mathbb{R}$, $0<\{\tilde{\alpha}+[V(0)]^\gamma\}/[V(0)]^\gamma<1$ and $\tilde{W}$ is monotone, concave and analytic on $\mathbb{R}^2_+$. Now if it so happens that $\tilde{\alpha}=\alpha[V(0)]^{\gamma-1}$, the objective function for the agent's problem (14) corresponding to $\tilde{W}$ is given by

$$\mathrm{E}\tilde{W}(c_1,rx)=\pi[f(c_1)V(Rx)]^\gamma. \tag{21}$$

Since (20) and (21) differ only by an increasing monotonic transform, they both yield the same consumption–asset demand correspondence. Hence it will be impossible to distinguish between W and $\tilde{W}$ even though they are not affinely equivalent NM indices.

We shall now show that although additive separability can *not* be dispensed with even when $\preccurlyeq$ is assumed to be NM representable, stationarity can indeed be.

Theorem 6. Suppose that Assumption 9 holds, and (U,V) and $(\tilde{U},\tilde{V})$ satisfy Assumptions 2 and 10. Further, let the representation of time preferences take the (ordinally) additively separable form (16). Then corresponding, respectively, to (U,V) and $(\tilde{U},\tilde{V})$ will be the two-period NM indices $W(c_1,c_2)=w(c_1)+V(c_2)$ and $\tilde{W}(c_1,c_2)=\tilde{w}(c_1)+\tilde{V}(c_2)$. If the same correspondence ξ is generated by W and $\tilde{W}$, then

$$w=a+b\tilde{w},\qquad b>0,$$

$$V=\alpha+\beta\tilde{V},\qquad \beta>0.$$

Proof. Again we shall only sketch the argument. The necessary first-order conditions for a maximum are

$$w'(c_1) = \lambda q, \qquad \mathrm{E} r V'(rx) = \lambda p. \tag{22}$$

(Sufficiency is assumed — see Remark 1.) Without loss of generality, we may assume $V'(0)=1$. Then (22) can be used to recover $w'(0)$. An argument along the lines of the first part of the proof of Theorem 5 can then be used to recover the derivatives of all orders of w at 0. Since w is assumed to possess an analytic extension on $\mathbb{R}_+$, this is equivalent to recovering the function w everywhere. To recover V, it will similarly suffice to recover $V^{(k)}(0)$ for $k \geqq 2$. But from eq. (22),

$$\mathrm{E} r V'(rx) = \frac{p}{q} w'(c_1). \tag{23}$$

Since the right-hand side is an observable function, differentiating (23) k times with respect to x yields $V^{(k)}(0)$ provided $\mathrm{E} r^k$ is well-defined and non-zero. But this is evident. Q.E.D.

Remark 9. As noted in section 3 the assumption of risk preference independence is not essential for recoverability in the case of the consumption/portfolio problem. It is important to recognize that this, however, is not the case for the single asset consumption/savings problem. Thus, for instance, it is easy to see that the additive separability of W hypothesized in Theorem 6 implies risk preference independence.

5. Conclusion

In addition to the possible relaxation of assumptions on the period two conditional NM index used in proving recoverability especially where no riskless asset is assumed, a number of questions open for further research are suggested by our analysis.

- —Can the consumption–asset demand correspondence be employed to reveal information about the joint asset return distribution?
- —What are the properties of the consumption–asset demand correspondence that characterize $\preccurlyeq$ being OCE versus NM representable? Furthermore, what properties of the correspondence characterize $\preccurlyeq$ being *conditionally* NM representable?
- —How do the conditions for recoverability carry over to the case in which it is the aggregate, as opposed to individual, consumption and asset demands that are observable? What properties of the individual agent's preferences have observable implications for aggregate behavior in the

consumption/portfolio or consumption/savings problem? For example, does the OCE/NM distinction have any observable implications for aggregate behavior?

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