

Approximate Aggregation under Uncertainty*

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For a collection of agents with von Neumann–Morgenstern preferences, a price-independent income distribution, and identical probability beliefs, there exists a von Neumann–Morgenstern approximate aggregator. The risk tolerance of the approximate aggregator is equal to the sum of the individual agent risk tolerances at prices which yield constant, “risk-free”, contingent consumption. The application of the approximate aggregator to standard asset pricing models in finance is discussed briefly. *Journal of Economic Literature* Classification Numbers: 022, 026, 521. © 1986 Academic Press, Inc.

1. INTRODUCTION

The problem of aggregation concerns the possibility of rationalizing aggregate demand by a utility function. When it exists, this utility function, or a fictitious agent possessing it, is termed an “aggregator.”¹ The construct of an aggregator has been employed in many different areas in economics. In international trade, the aggregator, or community indifference map, is used to derive market demand functions.² When income is optimally distributed, the aggregator is interpreted as a social welfare function.³ In financial economics, the aggregator is often used to derive market valuation expressions for financial securities; standard versions of the capital asset pricing model satisfy the requisite conditions for the existence of an aggregator.⁴

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¹ Samuelson [21].

² See, for example, Chipman [3].

³ Chipman and Moore [5].

⁴ See, for example, Rubinstein [20].

Despite this interest in the existence of an aggregator, most of the results to date have been negative.⁵ Two exceptions are the special cases of essentially identical homothetic (or quasi-homothetic) preferences^{6,7} and of diverse homothetic preferences with a fixed (i.e., price-independent) distribution of income.⁸ Furthermore, homotheticity has been shown to be not only sufficient for aggregation but essentially necessary as well.⁹

While most of this work on aggregation posits a deterministic setting, the issue can also be raised in risky settings such as the standard one-period portfolio allocation problem. Consider the demand for contingent commodities by agents with identical probabilistic beliefs and continuous, monotone, and concave expected utility functions. Then, paralleling the certainty case, aggregation is possible when individual agents possess essentially identical, homothetic (or quasi-homothetic) von Neumann–Morgenstern preferences,¹⁰ or diverse, homothetic von Neumann–Morgenstern preferences with a price-independent distribution of income.

In the framework of choice under uncertainty, this problem acquires an additional dimension. Even if an aggregator exists, the aggregator need not be von Neumann–Morgenstern. Homotheticity requires that the cardinal utility index of each agent exhibit constant relative risk aversion.¹¹ If the degree of relative risk aversion varies across agents, then although an aggregator exists (assuming a price-independent income distribution) the aggregator is not additively separable and hence not von Neumann–Morgenstern. Moreover, the aggregator can be identified only indirectly—a closed form expression for the utility function is not known.¹²

⁵ It was Sonnenschein's [23] original argument that in the absence of further restrictions on individual preferences, aggregate excess demand need satisfy only homogeneity and Walras' law. Debreu [6] gave a definitive version of the argument, which was elaborated on in Geanakoplos and Polemarchakis [8], among others, and in Polemarchakis [15] for the case of additively separable preferences. Polemarchakis [17] and Sonnenschein [24] partially extended the argument to the case of demand functions in which income varies independently of prices. Hildenbrand [11], on the other hand, derived restrictions on aggregate demand by imposing restrictions on the distribution of income.

⁶ Preferences which are representable by a continuous utility function that is ordinaly homogeneous of degree one are said to be homothetic. The Engel curves (income expansion paths) associated with homothetic preferences are rays through the origin. If the Engel curves are lines but not necessarily through the origin, preferences are said to be quasi-homothetic.

⁷ Chichilnisky and Heal [2] and Gorman [9, 10].

⁸ See Eisenberg [7] and Chipman [4].

⁹ Jerisson [12], Polemarchakis [16], and Schafer [22].

¹⁰ See Milne [14] and Rubinstein [20].

¹¹ Pollak [18].

¹² This problem has, of course, an analogue under certainty: If individual agents have preferences which are representable by constant elasticity of substitution utility functions but the elasticity of substitution differs across agents, then although an aggregator exists it does not have an additively separable representation.

Our goal in this paper is to investigate the degree to which the aggregate demand of a collection of von Neumann–Morgenstern agents can be approximated by the demand of a single von Neumann–Morgenstern agent. Such an *approximate aggregator* is of interest for a number of reasons:

- (1) Expected utility maximization is frequently identified with “rationality” in single-period allocation problems under uncertainty.
- (2) Interesting properties of aggregate demand, and of the objective function of the *exact* aggregator when it exists, are revealed by comparison with the demand and utility function of the approximate aggregator.
- (3) Tractability and estimation: The applications mentioned above require not just knowledge that an aggregator exists, when this is the case, but knowledge of the aggregator itself or at least of a good approximation. Unless the aggregator, or approximate aggregator, takes a sufficiently simple form, it is not possible to see how individual preferences are being aggregated and the resulting impact on aggregate demand.

Following preliminaries (in Section 2) we demonstrate (in Section 3) that, assuming individual agents possess identical probabilistic beliefs, an approximate aggregator can be constructed which is von Neumann–Morgenstern and has risk tolerance equal to the sum of the risk tolerances of the individual agents at prices which yield constant (“risk-free”) contingent consumption or, equivalently, has relative risk aversion equal to the income-weighted harmonic mean of the relative risk aversions of the individual agents. We explore several criteria of accuracy for the approximation. The approximation is exact locally (some may prefer the term infinitesimally); its global behavior is characterized by means of asymptotic results and error bounds. Subsequently (in Section 4) we extend parts of the analysis to the case of heterogeneous beliefs. We conclude (in Section 5) with a discussion of the application of our results to asset pricing models in finance and a discussion of the extension to the case of price-dependent income distributions. Proofs are given in the Appendix.

2. PRELIMINARIES

States of nature, or, equivalently, primitive securities or commodities, are indexed by $i, j, k = 1, \dots, n$ and individual agents by $h = 1, \dots, m$. The expected utility function of agent h is given by

$$\phi_h(x_h) = \sum_{i=1}^n \pi_{hi} u_h(x_{hi}),$$

where $\pi_h = (\dots, \pi_{hi}, \dots)$ is a strictly positive probability vector and u_h is a twice continuously differentiable, strictly increasing ($u'_h > 0$) and strictly concave ($u''_h < 0$) cardinal utility index defined on the state contingent consumption domain. The relative risk aversion function ρ_h is defined by¹³

$$\rho_{hi} = \rho_h(x_{hi}) = -x_{hi} u''_h(x_{hi}) / u'_h(x_{hi}),$$

and the risk tolerance r_h is defined by¹⁴

$$r_{hi} = r_h(x_{hi}) = -u'_h(x_{hi}) / u''_h(x_{hi}).$$

Aggregate income is normalized to equal unity; agent h receives the constant share $\delta_h > 0$ of aggregate income. Prices are strictly positive vectors $p = (\dots, p_i, \dots)$. The demand function of agent h , $x_h(p) = (\dots, x_{hi}(p), \dots)$, is thus the solution to the optimization problem of maximizing the objective function $\phi_h(x_h)$ subject to the budget constraint

$$\sum_{i=1}^n p_i x_{hi} = \delta_h.$$

We restrict attention to prices for which a solution to the optimization problem of each individual agent exists, is unique, and is characterized by the first-order conditions. It follows that the derivative of agent h 's demand for commodity i with respect to the price of another commodity j is given by

$$d_{hij}(p) = (r_{hi}(p) r_{hj}(p) - r_{hi}(p) x_{hj}(p)) \left(\sum_{k=1}^n p_k r_{hk}(p) \right)^{-1}, \quad i \neq j,$$

and

(1)

$$d_{hii}(p) = -\frac{r_{hi}(p)}{p_i} + (r_{hi}(p)^2 - r_{hi}(p) x_{hi}(p)) \left(\sum_{k=1}^n p_k r_{hk}(p) \right)^{-1}.$$

(See the Appendix for the derivation of these expressions.)

The aggregate demand function, indexed by A , $x_A(p) = (\dots, x_{Ai}(p), \dots)$, is

$$x_A(p) = \sum_{h=1}^m x_h(p)$$

and its derivatives satisfy

$$d_{Aij}(p) = \sum_{h=1}^m d_{hij}(p).$$

¹³ Arrow [1] and Pratt [19].

¹⁴ Wilson [25].

We seek an approximate aggregator, indexed by \hat{A} , with cardinal utility index $u_{\hat{A}}$, probabilistic beliefs $\pi_{\hat{A}} = (\dots, \pi_{\hat{A}i}, \dots)$, and hence the expected utility function

$$\phi_{\hat{A}}(x_{\hat{A}}) = \sum_{i=1}^n \pi_{\hat{A}i} u_{\hat{A}}(x_{\hat{A}i}).$$

The demand function of the approximate aggregator is derived subject to the aggregate budget constraint

$$\sum_{i=1}^n p_i x_{\hat{A}i} = 1.$$

The derivatives satisfy, as in the case of the individual agents,

$$d_{\hat{A}ij}(p) = (r_{\hat{A}i}(p) r_{\hat{A}j}(p) - r_{\hat{A}i}(p) x_{\hat{A}j}(p)) \left(\sum_{k=1}^n p_k r_{\hat{A}k}(p) \right)^{-1}, \quad i \neq j,$$

and

$$d_{\hat{A}ii}(p) = -\frac{r_{\hat{A}i}(p)}{p_i} + (r_{\hat{A}i}(p)^2 - r_{\hat{A}i}(p) x_{\hat{A}i}(p)) \left(\sum_{k=1}^n p_k r_{\hat{A}k}(p) \right)^{-1},$$

where $r_{\hat{A}}$ is the risk tolerance of the approximate aggregator.

Of interest is the special case of the cardinal utility index

$$u_h(x_{hi}) = \frac{1}{\alpha_h} (x_{hi})^{\alpha_h}, \quad \alpha_h < 1,$$

which displays constant relative risk aversion

$$\rho_h = 1 - \alpha_h$$

or, equivalently, constant elasticity of substitution

$$\eta_h = \frac{1}{1 - \alpha_h},$$

and leads to homothetic preferences represented by the homogeneous expected utility function

$$\phi_h(x_h) = \frac{1}{\alpha_h} \sum_{i=1}^n \pi_{hi} (x_{hi})^{\alpha_h}.$$

It follows that the demand function $x_h(p) = (\dots, x_{hi}(p), \dots)$ is given componentwise by

$$x_{hi}(p) = \delta_h \left(\frac{\pi_{hi}}{p_i} \right)^{1/\rho_h} / g_h(p),$$

where

$$g_h(p) = \sum_{k=1}^n \left(\frac{\pi_{hk}}{p_k} \right)^{1/\rho_h} p_k,$$

$$d_{hij}(p) = \frac{1}{\delta_h} \left(\frac{1}{\rho_h} - 1 \right) x_{hi}(p) x_{hj}(p), \quad i \neq j,$$

and

$$d_{hii}(p) = -\frac{1}{\rho_h} \frac{x_{hi}(p)}{p_i} + \frac{1}{\delta_h} \left(\frac{1}{\rho_h} - 1 \right) x_{hi}(p)^2.$$

Approximate aggregators with the same form of cardinal utility

$$u_{\bar{A}}(x_{\bar{A}i}) = \frac{1}{\alpha_{\bar{A}}} (x_{\bar{A}i})^{\alpha_{\bar{A}}}, \quad \alpha_{\bar{A}} < 1,$$

displaying constant relative risk aversion

$$\rho_{\bar{A}} = 1 - \alpha_{\bar{A}}$$

or, equivalently, constant elasticity of substitution

$$\eta_{\bar{A}} = \frac{1}{1 - \alpha_{\bar{A}}},$$

lead to the expected utility function

$$\phi_{\bar{A}}(x_{\bar{A}}) = \frac{1}{\alpha_{\bar{A}}} \sum_{i=1}^n \pi_{\bar{A}i} (x_{\bar{A}i})^{\alpha_{\bar{A}}}.$$

It follows that the demand function $x_{\bar{A}}(p) = (\dots, x_{\bar{A}i}(p), \dots)$ is given componentwise by

$$x_{\bar{A}i}(p) = \left(\frac{\pi_{\bar{A}i}}{p_i} \right)^{1/\rho_{\bar{A}}} \Big/ g_{\bar{A}}(p),$$

where

$$g_{\bar{A}}(p) = \sum_{k=1}^n \left(\frac{\pi_{\bar{A}k}}{p_k} \right)^{1/\rho_{\bar{A}}} p_k,$$

$$d_{\bar{A}ij}(p) = \left(\frac{1}{\rho_{\bar{A}}} - 1 \right) x_{\bar{A}i}(p) x_{\bar{A}j}(p), \quad i \neq j,$$

and

$$d_{\bar{\lambda}ii}(p) = -\frac{1}{\rho_{\bar{\lambda}}} \frac{x_{\bar{\lambda}i}(p)}{p_i} + \left(\frac{1}{\rho_{\bar{\lambda}}} - 1\right) x_{\bar{\lambda}i}(p)^2.$$

Returning to the general case, we introduce the following definition:

DEFINITION. An approximate aggregator displays the 0th-order property at prices p^* if

$$x_{\bar{\lambda}i}(p^*) = x_{\lambda_i}(p^*), \quad i = 1, \dots, n.$$

The approximate aggregator displays the 1st-order property at prices p^* if

$$d_{\bar{\lambda}ij}(p^*) = d_{\lambda_{ij}}(p^*), \quad i, j = 1, \dots, n.$$

The 0th-order property is easily satisfied. For the 1st-order property on the other hand, if n is large, the number of conditions to be satisfied can far exceed the number of parameters available. In the next section we show that, under the assumption of identical probabilistic beliefs, these conditions can indeed be met, however, and stronger ones as well.

3. IDENTICAL BELIEFS

Suppose that the probabilistic beliefs of the individual agents coincide, i.e., $\pi_h = (\dots, \pi_{hi}, \dots) = \pi$, $h = 1, \dots, m$, and consider the possibility of approximate aggregation at prices colinear with the probability vector π , i.e., $p^* = (1/q)\pi$, for some $q > 0$. At such prices, all agents equate their levels of consumption across states of nature. Hence,

$$\begin{aligned} x_{hi}(p^*) &= q\delta_h, \\ d_{hij}(p^*) &= q(r_h(p^*) - q\delta_h), \quad i \neq j, \end{aligned}$$

and

$$d_{hii}(p^*) = -\frac{r_h(p^*)}{p_i^*} + q(r_h(p^*) - q\delta_h).$$

It follows that any approximate aggregator satisfies the 1st-order property at p^* provided his risk tolerance at p^* is the sum of the risk tolerances of the individual agents at p^* .

$$r_{\bar{\lambda}}(p^*) = \sum_{h=1}^m r_h(p^*). \tag{2}$$

For example, the approximate aggregator with $\pi_{\bar{A}} = \pi$ and constant relative risk aversion $\rho_{\bar{A}}$ equal to the income-weighted harmonic mean of the relative risk aversion of the individual agents at p^* ,

$$\rho_{\bar{A}} = \left(\sum_{h=1}^m \delta_h \rho_h (p^*)^{-1} \right)^{-1}, \quad (3)$$

does indeed satisfy the 1st-order property at p^* .

We have thus demonstrated the following:

PROPOSITION 1. *For a collection of agents with von Neumann–Morgenstern preferences, a price-independent income distribution, and identical probability beliefs, there exists a von Neumann–Morgenstern approximate aggregator which displays the 0th-order and 1st-order properties at p^* and has risk tolerance (relative risk aversion) equal to the sum of the individual agent risk tolerances (income-weighted harmonic mean of the relative risk aversion of the individual agents).¹⁵*

Remark 1. Observe that the risk tolerance function $r_{\bar{A}}$ determines (up to a positive affine transformation by integration as, for instance, in Pratt [19]) the corresponding cardinal utility index $u_{\bar{A}}$ of the approximate aggregator.

Remark 2. The above does *not* require the individual agents to exhibit constant relative risk aversion and thus does not require the existence of an exact, albeit non-von Neumann–Morgenstern, aggregator.

Remark 3. It may seem puzzling that the risk tolerance of the approximate aggregator is not an income-weighted average of the risk tolerance of the individual agents: an individual with zero income may influence aggregate behavior. But this is to be expected if preferences are well defined for negative state contingent consumption, since we have imposed no restrictions on short sales. Of course, the demand and risk aversion of the individual agents will not be independent of income.

Remark 4. At first glance, it may seem that the expressions for $r_{\bar{A}}(p^*)$ and $\rho_{\bar{A}}$, (2) and (3), are incompatible in that the income distribution, $\{\delta_h\}$, appears in the latter, but not in the former. However, this is not the case, as the $\{\delta_h\}$ is present implicitly in the set of functions $\{r_i(p^*)\}$.

We next examine briefly the special case of two agents, $h = 1, 2$, and two states of nature, $i = 1, 2$. We suppose furthermore that both agents display constant relative risk aversion, ρ_h , $h = 1, 2$, and hence an exact, but not necessarily von Neumann–Morgenstern, aggregator, A , exists. Two of the

¹⁵ We are indebted to the Associate Editor for helpful suggestions which significantly generalized an earlier version of this result.

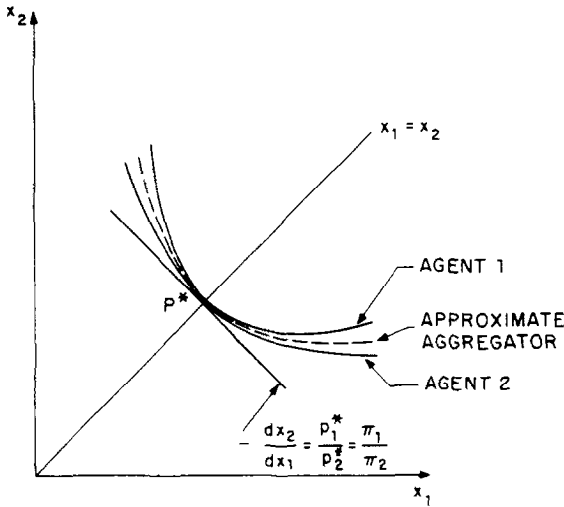


FIGURE 1

indifference curves in Fig. 1 correspond to the expected utility functions ϕ_h , $h = 1, 2$. At $p^* = (1/q)\pi$ both agents select demands along the 45° ray and their indifference curves share a common tangent with slope π_1/π_2 . The indifference curve of the approximate aggregator constructed above also shares the same tangent line along the diagonal. Furthermore, since the elasticity of substitution $\eta_{\bar{A}} = 1/\rho_{\bar{A}}$ is by construction the income-weighted average of the elasticities of substitution of the individual agents, the level curves of $\phi_{\bar{A}}$ are intermediate in curvature between the level curves of the individual agents.

Returning to the general case, is there any particular reason to center the approximation at prices p^* colinear with the probability vector π ? At such prices, the individual agents and the economy as a whole equate consumption across states of nature; thus p^* , in effect, constitutes the special case of certainty, and small deviations from p^* generate small differences in aggregate demand for the two commodities and hence “small risks” (in much the same sense as in the classic Arrow–Pratt work on risk aversion). When an exact aggregator exists, even if it fails to be von Neumann–Morgenstern, risk aversion can be identified with the convexity of the contingent commodity indifference curves and hence its curvature can be used as a measure of the degree of risk aversion.¹⁶ It then follows

¹⁶ As ϕ_A fails, in general, to be an expected utility function, it is necessary to adopt a general notion of risk aversion such as that developed in Yaari [26]. The argument proceeds by deriving a local measure of risk aversion from the acceptance frontier (indifference curve) in state contingent consumption space; for expected utility functions, the Yaari measure coincides with the Arrow–Pratt measure of absolute risk aversion. A straightforward modification yields the analogous result for relative risk aversion.

from the 1st-order property that the exact and the approximate aggregator display the same degree of risk aversion at p^* .

We conjecture but we have not been able to prove analytically that, if individual agents display constant relative risk aversion, the approximate aggregator constructed above is the only approximate aggregator with constant relative risk aversion at p^* or any other price vector.

To evaluate further the accuracy of the approximation we first examine more closely its local behavior at p^* for the case of constant relative risk aversion agents. The difference between the second derivative of the aggregate demand function and the second derivative of the demand of the approximate aggregator is given by

$$\begin{aligned}
 & d_{Aijk}(p^*) - d_{\hat{A}ijk}(p^*) \\
 &= 2q^3 \left\{ \sum_{h=1}^m \delta_h (\rho_h^{-1} - 1)^2 - \left[\sum_{h=1}^m \delta_h \rho_h^{-1} - 1 \right]^2 \right\}, \quad i \neq j \neq k, \\
 & d_{Aijj}(p^*) - d_{\hat{A}ijj}(p^*) \tag{4} \\
 &= q^3 \left(2 - \frac{1}{\pi_i} \right) \left\{ \sum_{h=1}^m \delta_h (\rho_h^{-1} - 1)^2 - \left[\sum_{h=1}^m \delta_h \rho_h^{-1} - 1 \right]^2 \right\}, \quad i \neq j, \\
 & d_{Aiii}(p^*) - d_{\hat{A}iii}(p^*) \\
 &= q^3 \left(2 - \frac{3}{\pi_i} + \frac{1}{\pi_i^2} \right) \left\{ \sum_{h=1}^m \delta_h (\rho_h^{-1} - 1)^2 - \left[\sum_{h=1}^m \delta_h \rho_h^{-1} - 1 \right]^2 \right\}.
 \end{aligned}$$

(See the Appendix for the derivation of these equations.)

It is worth noting that the expression in brackets can be viewed as the (weighted) variance of the quantities ρ_h^{-1} about the value $\rho_h = 1$; when ρ_h does not vary with h , the expression vanishes.

We now attempt to shed some light on the global behavior of the approximation, i.e., for prices p not necessarily close to p^* , we maintain the assumption that the individual agents as well as the approximate aggregator display constant relative risk aversion.

We consider first asymptotic properties of the (generally nonconstant) elasticity of substitution for consumption in states i and j , $i \neq j$:

$$\eta_{Aij} = 1 + (d_{Aij}/x_{Ai}x_{Aj}),$$

where this simplified expression for the elasticity of substitution follows from the homogeneity of ϕ_A . Note that the elasticity of substitution, η_{Aij} , is a measure of the curvature of the aggregator's indifference curve and hence of the degree of risk aversion with respect to consumption in states i and j . The idea is to show that η_{Aij} does not depart too drastically from the constant elasticity of substitution of the approximate aggregator

$\eta_{\bar{A}} = \sum_{h=1}^m \delta_h \eta_h$, even for prices far from p^* . For simplicity, we assume that either $\rho_h < 1$ ($\eta_h > 1$; $\alpha_h > 0$) for all h or $\rho_h > 1$ ($\eta_h < 1$; $\alpha_h < 0$) for all h . Let

$$\eta_+ = \max_h \{\eta_h\}, \quad \eta_- = \min_h \{\eta_h\},$$

and assume these extrema are realized by unique agents indexed $h = +$ and $h = -$, respectively. We consider limits as the price vector $p \rightarrow \bar{p}$ with $\bar{p}_i = 0$, $\bar{p}_j > 0$.

We obtain the following:

PROPOSITION 2A. *For individual agents with constant relative risk aversion, expected utility preferences:*

(a) *If all $\eta_h > 1$ ($\rho_h < 1$), then*

$$\lim_{p \rightarrow \bar{p}} \eta_{Aij} = \eta_- (\rho_+^{-1}).$$

(b) *If all $\eta_h < 1$ ($\rho_h > 1$), then for $h > 2$*

$$\lim_{p \rightarrow \bar{p}} \eta_{Aij} = 1 + \frac{x_{+j}(p)}{\delta_+ x_{Aj}(\bar{p})} (\eta_+ - 1);$$

for $h = 2$

$$\lim_{p \rightarrow \bar{p}} \eta_{Aij} = \eta_+ (\rho_-^{-1}).$$

(The proof is given in the Appendix.)

Observe that if the agents' elasticities of substitution all exceed (are less than) that of the logarithmic utility index (having $\alpha = 0$), then asymptotically the economy as a whole exhibits the elasticity of substitution of the agent with the smallest (largest) elasticity of substitution, that is, the agent closest to the logarithmic case.¹⁷

Next we examine global behavior from a different point of view; we exhibit bounds on the growth of the error resulting from deriving aggregate demand using the expected utility function of the approximate aggregator $\phi_{\bar{A}}(x_{\bar{A}})$. Using commodity n as a reference, we measure the relative value of the demand of the approximate aggregator with respect to that of the exact aggregator by

$$e_i(p) = \frac{x_{\bar{A}i}(p)/x_{\bar{A}n}(p)}{x_{Ai}(p)/x_{An}(p)}, \quad i = 1, \dots, n - 1;$$

¹⁷ For more than two goods it can be shown that, as p approaches the boundary of the price simplex, every η_{Aij} approaches a finite constant; typically, these constants are rather complex.

the approximation is exact when $e_i = 1$. Let

$$\tau_i = \frac{\pi_i/p_i}{\pi_n/p_n}, \quad i = 1, \dots, n-1.$$

It then follows that:

PROPOSITION 2B.

- (a) $\tau_i^{(1/\rho_{\bar{\lambda}} - 1/\rho_+)} \leq e_i \leq \tau_i^{(1/\rho_{\bar{\lambda}} - 1/\rho_-)}, \quad \tau_i \leq 1,$
- (b) $\tau_i^{(1/\rho_{\bar{\lambda}} - 1/\rho_-)} \leq e_i \leq \tau_i^{(1/\rho_{\bar{\lambda}} - 1/\rho_+)}, \quad \tau_i \geq 1.$

(The proof is given in the Appendix.) Note that the bounds are exact at $p = p^*$, where $\tau = (\dots, \tau_i, \dots) = 1$.

Observe that if the individual agents have similar but not identical preferences, the powers $(1/\rho_{\bar{\lambda}} - 1/\rho_+)$ and $(1/\rho_{\bar{\lambda}} - 1/\rho_-)$ are near zero; hence, as p departs from p^* and thus τ from 1, the upper and lower bounds grow away from 1, but only slowly.

To conclude the case of identical beliefs, observe that exact aggregation is possible if the cardinal utility indices of individual agents are exponential functions

$$u_h(x_{hi}) = -e^{-\alpha_h x_{hi}}, \quad \alpha_h > 0,$$

displaying constant risk tolerance

$$r_h = \frac{1}{\alpha_h}.$$

Let

$$r_A = \sum_{h=1}^m r_h$$

or, equivalently,

$$\alpha_A = \left(\sum_{h=1}^m \alpha_h^{-1} \right)^{-1}$$

and consider the aggregator with cardinal utility index

$$u_A(x_{Ai}) = -e^{-\alpha_A x_{Ai}}.$$

The aggregate demand $x_A(p)$ clearly coincides with the demand of the aggregator for prices p colinear with the probability vector π . To see that

the aggregate demand coincides with the demand of the aggregator for all prices, it then suffices to observe that

$$d_{Aij}(p) = (r_A - x_{Aj}(p)) \left(\sum_{k=1}^n p_k \right)^{-1}, \quad i \neq j,$$

and

$$d_{Aii}(p) = -\frac{r_A}{p_i} + (r_A - x_{Ai}(p)) \left(\sum_{k=1}^n p_k \right)^{-1}.$$

Hence, the aggregate demand function satisfies the differential equation that characterizes the demand function of the aggregator; this completes the argument.¹⁸

Remark 5. The example of agents with diverse negative exponential, expected utility preferences raises an interesting issue. Had we not known that an exact aggregator exists, we probably would have chosen to approximate the economy with a constant relative risk aversion $\phi_{\hat{A}}$ satisfying the condition (3). As a general proposition, the error bounds and asymptotic properties derived above justify assuming $\phi_{\hat{A}}$ exhibits constant relative risk aversion. More fundamentally, however, the question can be raised whether for common (parametric) classes of von Neumann–Morgenstern representations the best approximation, appropriately defined, need be in the same class. Thus, for instance, if each agent in the economy is characterized by a different expected utility function in the HARA class widely employed in finance models, and an exact aggregator fails to exist or is not obtainable in closed form, then must the “best” *approximate* aggregator be in the same preference class?

4. DIVERSE BELIEFS

We now attempt to relax the assumption of identical beliefs across agents. Our results are limited, however to the case of only two agents ($m = 2$), both displaying constant relative risk aversion.

When the degrees of relative risk aversion of the two agents are distinct, the approximate aggregator with constant relative risk aversion

$$\rho_{\hat{A}} = \left(\sum_{h=1}^2 \delta_h \rho_h^{-1} \right)^{-1}$$

¹⁸ This observation was made by Lintner [13] under the joint assumptions of negative exponential cardinal utility and normal probability beliefs.

and probability beliefs $\pi_{\tilde{A}} = (\dots, \pi_{\tilde{A}i}, \dots)$ given by

$$\pi_{\tilde{A}i} = \prod_{h=1}^2 \pi_{hi}^{\delta_h \rho_{\tilde{A}} / \rho_h}$$

satisfies the 0th-order and 1st-order properties at $p^* = (\dots, p_i^*, \dots)$ with

$$p_i^* = \pi_{1i}^{\rho_2 / (\rho_2 - \rho_1)} \pi_{2i}^{\rho_1 / (\rho_1 - \rho_2)}.$$

The results on the asymptotic behavior extend directly to the present case. Analogous error bounds can also be derived, but they are more complex and depend on the divergence of the probability beliefs as well as of the preferences.

We conjecture, but we have not been able to prove, that the results generalize to an arbitrary number of agents.

The case of identical preferences but distinct probability beliefs presents an anomaly. Assume, for convenience, that $n = m = 2$. The previous results would suggest setting $\rho_{\tilde{A}} = \rho$, where $\rho = \rho_1 = \rho_2$. For $\rho \neq 1$, however, it can be shown that $\rho_A(p) \neq \rho$. In particular, the risk aversion of the exact aggregator is closer to that of the logarithmic cardinal utility function than either of the two agents are. (This assertion is demonstrated in the last section of the Appendix.)

5. APPLICATIONS AND EXTENSIONS

In this section we consider an application of our approximate aggregator and a possible extension of the analysis to a more general setting.

Aggregators have frequently been employed to characterize equilibrium in securities markets. Consider an economy extending over a single time period in which there are N states of nature and M assets ($N \geq M$), one of which is risk-free (denoted F). Individual agents have identical probability beliefs concerning the distribution of returns of the various assets.

As mentioned in the Introduction, if agents have identical, homothetic preferences, an aggregator exists and is von Neumann–Morgenstern. Then the first-order conditions for maximization of the aggregate expected utility function, $Eu_A(x)$, combined with the balance equations yield the relations¹⁹

$$\begin{aligned} E[u'_A(x)(R_j - R_F)] &= 0, & j = 1, \dots, M - 1, \\ E(R_j) &= R_F + \lambda k_j \sigma(R_j), & j = 1, \dots, M - 1, \end{aligned}$$

¹⁹ Rubinstein [20].

where E and σ are the expectation and standard deviation operators, respectively, R_j is the random rate of return of the risky asset j , R_F is the risk-free rate of return, $u'_A(x)$ is the marginal utility of random end-of-period consumption, k_j is the "correlation coefficient" between R_j and $-u'_A(x)$, and $\lambda \equiv \sigma(u'_A(x))/E(u'_A(x))$. Suppose now agents have diverse von Neumann–Morgenstern preferences with fixed income shares—equivalently colinear initial endowments. Then an exact aggregator need not exist and/or need not be von Neumann–Morgenstern. Yet, the cardinal utility index of the approximate aggregator $u_{\bar{A}}$ can be employed to yield approximate pricing relations.

Remark 6. When aggregation obtains under the assumption of essentially identical homothetic preferences, markets are effectively complete: The introduction of new assets, assuming $M < N$, does not affect either the prices of the existing assets or the state contingent consumptions of individual agents at equilibrium.²⁰ The new securities will, in fact, not be held by any agent because of the assumed homogeneity of the individual agents and the required zero net supply of the securities. In our framework, aggregation obtains even though individual agents may have diverse preferences. Although the introduction of new assets in our incomplete market setup still leaves the equilibrium price of existing securities unchanged (they are priced based on the objective function of the aggregator), the new securities may, however, be traded among the diverse agents in which case their introduction will lead to a Pareto improvement in the allocation of resources at equilibrium. The possibly adverse effects on individual agents of changes in the prices of existing securities are precluded by the existence of an (exact) aggregator as argued above.

Remark 7. The heterogeneity of individual agents has still an additional consequence: The prices of assets at equilibrium are not independent of the distribution of income. (See Remark 3 above.) This differs from the case of essentially identical individual agents in which the attitude toward risk of the aggregator is independent of the distribution of income across individuals. When individuals are diverse, the characterization of the aggregator, and hence asset prices, depend on the assumed income distribution.

Turning to extensions, preliminary work indicates that the assumption of fixed income shares can be relaxed, at least when the agents display constant relative risk aversion. Using the shares $\delta_h^* = \delta_h(p^*)$ which obtain at $p^* = (1/q)\pi$ for the case of identical beliefs allows us to define $\rho_{\bar{A}}$, the constant relative risk aversion of the approximate aggregator as above. The

²⁰ Rubinstein [20].

0th-order and 1st-order properties at p^* can then be demonstrated and global error bounds can be obtained. Thus, in two senses at least, aggregate demands are closely approximated by ϕ_{λ} .

APPENDIX: PROOFS

Equation (1) in Section 2

The demand function of individual h is derived as the solution to

$$\begin{aligned} & \text{Max}_{x_h} \sum_{i=1}^n \pi_{hi} u_h(x_{hi}), \\ & \text{s.t.} \sum_{i=1}^n p_i x_{hi} = \delta_h. \end{aligned}$$

The first-order necessary and sufficient conditions for an interior solution are

$$\begin{aligned} \pi_{hi} u'_h(x_{hi}) &= \lambda_h p_i, \quad i = 1, \dots, n, \\ \sum_{i=1}^n p_i x_{hi} &= \delta_h, \end{aligned}$$

for some $\lambda_h > 0$.

Totally differentiating the first-order conditions, we obtain

$$\begin{bmatrix} \pi_{h1} u''_{h1}, & \dots, & & -p_1 \\ & \ddots & & \vdots \\ & & 0 & -p_i \\ & & \pi_{hi} u''_{hi} & \vdots \\ 0 & & \vdots & -p_n \\ & & \pi_{hn} u''_{hn}, & -p_n \\ -p_1, \dots, & -p_i, \dots, & -p_n & 0 \end{bmatrix}^{-1} = \begin{bmatrix} S_{h11}, \dots, S_{h1n}, & -v_{h1} \\ \vdots & \vdots \\ S_{hn1}, \dots, S_{hnn}, & -v_{hn} \\ -v_{h1}, \dots, -v_{hn}, & \epsilon_h \end{bmatrix},$$

and

$$\frac{\partial x_{hi}}{\partial p_j} = \lambda_h s_{hij} - v_{hi} x_{hj}.$$

But a straightforward argument implies that

$$S_{hij} = \frac{-p_i p_j}{(\pi_{hi} u''_{hi})(\pi_{hj} u''_{hj})} \left(\sum_{k=1}^n \frac{p_k^2}{\pi_{hk} u''_{hk}} \right)^{-1}, \quad i \neq j;$$

$$S_{hii} = \frac{1}{\pi_{hi} u''_{hi}} - \frac{p_i^2}{(\pi_{hi} u''_{hi})^2} \left(\sum_{k=1}^n \frac{p_k^2}{\pi_{hk} u''_{hk}} \right)^{-1};$$

$$v_{hi} = \frac{p_i}{\pi_{hi} u''_{hi}} \left(\sum_{k=1}^m \frac{p_k^2}{\pi_{hk} u''_{hk}} \right)^{-1}.$$

Setting $r_{hi} = -u'_{hi}/u''_{hi} = -\lambda_h p_i/\pi_{hi} u''_{hi}$ and substituting yields Eq. (1) in the text.

The remaining arguments assume that each agent exhibits constant relative risk aversion. For convenience, we express all quantities in terms of $\eta = 1/\rho$.

Equation (4) in Section 3

For constant ρ_h we have

$$r_{hi} = x_{hi}/\rho_h = x_{hi}\eta_h = \eta_h \delta_h (\pi_i/p_i)^{\eta_h} g_h^{-1},$$

and so

$$d_{hij} = (x_{hi}x_{hj}\eta_h^2 - x_{hi}x_{hj}\eta_h) \left(\sum_k p_k x_{hk}\eta_h \right)^{-1}$$

$$= \delta_h (\eta_h - 1) (\pi_i \pi_j / p_i p_j)^{\eta_h} g_h^{-2}, \quad i \neq j,$$

$$d_{hii} = -x_{hi}\eta_h/p_i + (x_{hi}^2\eta_h^2 - x_{hi}^2\eta_h) \left(\sum_k p_k x_{hk}\eta_h \right)^{-1}$$

$$= -\delta_h \eta_h (\pi_i/p_i)^{\eta_h} p_i^{-1} g_h^{-1} + \delta_h (\eta_h - 1) (\pi_i/p_i)^{2\eta_h} g_h^{-2}.$$

Now, direct differentiation yields

$$d_{hijk} = 2\delta_h (\eta_h - 1)^2 (\pi_i \pi_j \pi_k / p_i p_j p_k)^{\eta_h} g_h^{-3}, \quad i \neq j \neq k \neq i,$$

$$d_{hijj} = -\delta_h (\eta_h - 1) \eta_h (\pi_i \pi_j / p_i p_j)^{\eta_h} p_i^{-1} g_h^{-2}$$

$$+ 2\delta_h (\eta_h - 1)^2 (\pi_i^2 \pi_j / p_i^2 p_j)^{\eta_h} g_h^{-3}, \quad i \neq j,$$

$$d_{hiii} = \delta_h \eta_h (\eta_h + 1) (\pi_i/p_i)^{\eta_h} p_i^{-2} g_h^{-1}$$

$$- 3\delta_h (\eta_h - 1) \eta_h (\pi_i/p_i)^{2\eta_h} p_i^{-1} g_h^{-2}$$

$$+ 2\delta_h (\eta_h - 1)^2 (\pi_i/p_i)^{3\eta_h} g_h^{-3}.$$

At p^* we have $g_h = \sum_k q^{\eta_h} \pi_k / q = q^{\eta_h - 1}$, and so

$$d_{Aijk} = 2q^3 \sum_h \delta_h (\eta_h - 1)^2, \quad i \neq j \neq k,$$

$$d_{Aij} = -(q^3/\pi_i) \sum_h \delta_h (\eta_h - 1) \eta_h + 2q^3 \sum_h \delta_h (\eta_h - 1)^2, \quad i \neq j,$$

$$d_{Aii} = (q^3/\pi_i^2) \sum_h \delta_h \eta_h (\eta_h + 1) - 3(q^3/\pi_i) \sum_h \delta_h (\eta_h - 1) \eta_h + 2q^3 \sum_h \delta_h (\eta_h - 1)^2.$$

Also, at p^*

$$\begin{aligned} d_{\bar{A}ijk} &= 2q^3(\eta_{\bar{A}} - 1)^2, \quad i \neq j \neq k \\ d_{\bar{A}ijj} &= -(q^3/\pi_i)(\eta_{\bar{A}} - 1) \eta_{\bar{A}} + 2q^3(\eta_{\bar{A}} - 1)^2, \quad i \neq j, \\ d_{\bar{A}iii} &= (q^3/\pi_i^2) \eta_{\bar{A}}(\eta_{\bar{A}} + 1) - 3(q^3/\pi_i)(\eta_{\bar{A}} - 1) \eta_{\bar{A}} + 2q^3(\eta_{\bar{A}} - 1)^2. \end{aligned}$$

The result now follows from observing that

$$\begin{aligned} &\sum_h \delta_h (\eta_h - 1) \eta_h - (\eta_{\bar{A}} - 1) \eta_{\bar{A}} \\ &= \left[\sum_h \delta_h (\eta_h - 1)^2 + \eta_{\bar{A}} \right] - [(\eta_{\bar{A}} - 1)^2 + \eta_{\bar{A}}] \\ &= \sum_h \delta_h (\eta_h - 1)^2 - \left[\sum_h \delta_h (\eta_h - 1) \right]^2 \end{aligned}$$

and

$$\begin{aligned} &\sum_h \delta_h (\eta_h + 1) \eta_h - (\eta_{\bar{A}} + 1) \eta_{\bar{A}} \\ &= \left[\sum_h \delta_h (\eta_h - 1)^2 + 3\eta_{\bar{A}} \right] - [(\eta_{\bar{A}} - 1)^2 + 3\eta_{\bar{A}}] \\ &= \sum_h \delta_h (\eta_h - 1)^2 - \left[\sum_h \delta_h (\eta_h - 1) \right]^2. \end{aligned}$$

Proposition 2A in Section 3

(a) All $\eta_h > 1$ ($\rho_h < 1$):

For all i, h , we have

$$\left[\frac{\pi_i}{p_i} \right]^{\eta_h} g_h^{-1} = \left[\sum_k \left[\frac{\pi_k}{\pi_i} \right]^{\eta_h} p_i^{\eta_h} p_k^{1-\eta_h} \right]^{-1}.$$

Using this equation, we have, for $i \neq j$,

$$d_{hij} = \delta_h (\eta_h - 1) \left\{ \left[\sum_k \left[\frac{\pi_k}{\pi_i} \right]^{\eta_h} p_i^{\eta_h} p_k^{1-\eta_h} \right] \left[\sum_v \left[\frac{\pi_v}{\pi_j} \right]^{\eta_h} p_j^{\eta_h} p_v^{1-\eta_h} \right] \right\}^{-1}$$

The terms in the first summation for $k \neq i$ go to zero faster than the term for $k = i$, since $\eta_h > 1$, as $p \rightarrow \bar{p}$. In the second summation all terms go to constants except for $v = i$, and that term $\rightarrow \infty$. Hence, as $p \rightarrow \bar{p}$,

$$d_{hij} \rightarrow \delta_h (\eta_h - 1) \left[\left[\frac{\pi_i}{\pi_j} \right]^{\eta_h} p_j^{\eta_h} p_i^{2 - \eta_h} \right]^{-1}.$$

Similarly,

$$\begin{aligned} x_{Ai} x_{Aj} &= \sum_{h,l} \delta_h \delta_l \left\{ \left[\sum_k \left[\frac{\pi_k}{\pi_i} \right] p_i^{\eta_l} p_k^{1 - \eta_l} \right] \left[\sum_v \left[\frac{\pi_v}{\pi_j} \right] p_j^{\eta_h} p_v^{1 - \eta_h} \right] \right\}^{-1} \\ &= \sum_{h,l} \delta_h \delta_l \left[\left[\frac{\pi_l}{\pi_j} \right]^{\eta_h} p_j^{\eta_h} p_i^{2 - \eta_h} \right]^{-1} = \sum_h \delta_h \left[\left[\frac{\pi_i}{\pi_j} \right]^{\eta_h} p_j^{\eta_h} p_i^{2 - \eta_h} \right]^{-1}. \end{aligned}$$

Now, in this summation, the term with $\eta_h = \eta_-$ eventually dominates the others, since $p_i^{2 - \eta_-}$ either goes to zero faster (if $\eta_- \leq 2$) or to ∞ slower (if $\eta_- > 2$) than the other $p_i^{2 - \eta_h}$. The same is true of d_{Aij} . Hence,

$$\eta_{Aij} - 1 + \frac{\delta_- \left[\left[\frac{\pi_i}{\pi_j} \right]^{\eta_-} p_j^{\eta_-} p_i^{2 - \eta_-} \right]^{-1}}{\delta_- \left[\left[\frac{\pi_i}{\pi_j} \right]^{\eta_-} p_j^{\eta_-} p_i^{2 - \eta_-} \right]^{-1}} (\eta_- - 1) = \eta_- ,$$

where δ_- is the δ_h corresponding to η_- .

(b) All $\eta_h < 1$ ($\rho_h > 1$):

Writing

$$g_h(p) = \sum_k \pi_k^{\eta_h} p_k^{1 - \eta_h},$$

it is clear that, as $p \rightarrow \bar{p}$, g goes to a finite constant. Also, we have

$$x_{Ai} = \sum_h \delta_h \left[\frac{\pi_i}{p_i} \right]^{\eta_h} g_h^{-1}.$$

As $p \rightarrow \bar{p}$, all terms $\rightarrow \infty$, and the one having $\eta_h = \eta_+$ dominates the others. Similar remarks apply to d_{Aij} . Thus,

$$\begin{aligned} \eta_{Aij} &\rightarrow 1 + \left[\frac{\delta_+ (\pi_i/p_i)^{\eta_+} g_+^{-1}}{\delta_+ (\pi_i/p_i)^{\eta_+} g_+^{-1}} \right] \left[\frac{(\pi_j/p_j)^{\eta_+} g_+^{-1}}{x_{Aj}} \right] (\eta_+ - 1) \\ &= 1 + (x_{+j}/\delta_+ x_{Aj})(\eta_+ - 1). \end{aligned}$$

For $n = 2$, we have for all h $x_{hj}/\delta_h x_{Aj} \rightarrow 1$. Hence,

$$\eta_{Aij} \rightarrow 1 + (\eta_+ - 1) = \eta_+.$$

Proposition 2B in Section 3

We may write

$$x_{hi} = \delta_h (\pi_i/p_i)^{\eta_h} g_h^{-1} = x_{hn} \tau_i^{\eta_h}, \quad \text{for all } h, i.$$

Defining $w_h = x_{hn}/x_{An}$, we have

$$x_{Ai}/x_{An} = \sum_h w_h \tau_i^{\eta_h}.$$

We now apply a version of the intermediate value theorem (Zipkin [27]) to conclude that there exist functions $\eta_i(p)$ such that

$$\eta_- \leq \eta_i(p) \leq \eta_+, \quad \text{for all } i \text{ and } p,$$

and

$$x_{Ai}/x_{An} = \tau_i^{\eta_i(p)}.$$

Since

$$x_{\lambda i}/x_{\lambda n} = \tau_i^{\eta_i^\lambda},$$

we have

$$e_i = \tau_i^{\eta_i^\lambda - \eta_i(p)}.$$

The bounds now follow by replacing $\eta_i(p)$ by its worst-case value, either η_- (ρ_+) or η_+ (ρ_-) depending on τ_i .

Diverse Beliefs and Homogeneous Preferences—Section 4

As suggested in the text, assume there are two constant relative risk averse agents. Then we have

$$\eta_A = 1 + \frac{\partial x_{A1}/\partial p_2}{x_{A1} x_{A2}} = 1 + (\eta - 1) \times \left\{ \frac{\delta_1 \left[\frac{\pi_{11} \pi_{12}}{p_1 p_2} \right]^\eta g_1^{-2} + \delta_2 \left[\frac{\pi_{21} \pi_{22}}{p_1 p_2} \right]^\eta g_2^{-2}}{\left[\delta_1 \left[\frac{\pi_{11}}{p_1} \right]^\eta g_1^{-1} + \delta_2 \left[\frac{\pi_{21}}{p_1} \right]^\eta g_2^{-1} \right] \left[\delta_1 \left[\frac{\pi_{12}}{p_2} \right]^\eta g_1^{-1} + \delta_2 \left[\frac{\pi_{22}}{p_2} \right]^\eta g_2^{-1} \right]} \right\}.$$

Define

$$f_h = \left[\frac{\pi_{h1}}{p_1} \right]^\eta p_1 g_h^{-1}.$$

Using the definition of g_h , we have

$$1 - f_h = \left[\frac{\pi_{h2}}{p_2} \right]^\eta p_2 g_h^{-1}.$$

Multiplying the top and bottom of the fraction above by $p_1 p_2$, we have

$$\eta_A = 1 + (\eta - 1) \left\{ \frac{\delta_1 f_1 (1 - f_1) + \delta_2 f_2 (1 - f_2)}{[\delta_1 f_1 + \delta_2 f_2][\delta_1 (1 - f_1) + \delta_2 (1 - f_2)]} \right\}.$$

Now, let N be the numerator of this fraction and D the denominator. Using $\delta_h^2 = \delta_h - \delta_1 \delta_2$,

$$\begin{aligned} D &= \delta_1^2 f_1 (1 - f_1) + \delta_2^2 f_2 (1 - f_2) + \delta_1 \delta_2 [f_1 (1 - f_2) + f_2 (1 - f_1)] \\ &= N + \delta_1^2 \delta_2 [f_1 (1 - f_2) - f_1 (1 - f_1) - f_2 (1 - f_1) + f_2 (1 - f_1)] \\ &= N + \delta_1 \delta_2 (f_1 - f_2)^2. \end{aligned}$$

Moreover,

$$f_h = \left[1 + \left[\frac{\pi_{h2}}{\pi_{h1}} \right]^\eta \left[\frac{p_2}{p_1} \right]^{1-\eta} \right]^{-1};$$

so $\pi_{1i} \neq \pi_{2i}$ implies $f_1 \neq f_2$ for all p , hence

$$0 < N/D < 1.$$

Thus, if $\eta - 1 > 0$ ($\rho < 1$), $\eta_A < \eta$ ($\rho_A > \rho$), while if $\eta - 1 < 0$ ($\rho > 1$), $\eta_A > \eta$ ($\rho_A < \rho$), as asserted.

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