

Shortfall Aversion*

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Abstract

Shortfall aversion reflects the higher utility loss of a spending cut from a reference point than the utility gain from a similar spending increase, in the spirit of Prospect Theory's loss aversion. This paper posits a model of utility of spending scaled by a function of past peak spending, called target spending. The discontinuity of the marginal utility at the target spending corresponds to shortfall aversion. According to the closed-form solution of the associated spending-investment problem, (i) the spending rate is constant and equals the historical peak for relatively large values of wealth/target; and (ii) the spending rate increases (and the target with it) when that ratio reaches its model-determined upper bound. These features contrast with traditional Merton-style models which call for spending rates proportional to wealth. A simulation using the 1926-2012 realized returns suggests that spending of the very shortfall averse is typically increasing and very smooth.

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1 Introduction

In many circumstances spending adheres to its past peak or, if savings allow, gradually exceeds it, thereby establishing new peaks. Cutting spending, however, leads to a disappointment. Intuitively, this disappointment, given the size of the shortfall, is greater than the pleasure associated with a similar increase in spending relative to the same peak in past spending. This greater sensitivity to shortfall than to gain is familiar – it is loss aversion in consumption, referred to as *shortfall aversion*. It is the focus of this paper.

The static work on Prospect Theory (Kahneman and Tversky, 1979) inspires this paper’s dynamic model of shortfall aversion, suggesting that the present model is of an individual’s (or a household’s) choice of spending, saving and, given the amount of saving, portfolio selection.

The model also applies to the endowment of a university or a foundation that funds various programs with long-term spending commitments to employees and beneficiaries. Guided by their plans and expectations, these employees and beneficiaries in turn make their spending commitments. Thus, there are layers of reliance on the spending level. A spending cut leads to a waste of resources and a utility loss which is substantially larger than the utility gain associated with an increase in spending of a similar size.

Therefore it is reasonable, even imperative, to model the benefit that an individual or foundation derives from spending as not merely increasing in spending but as increasing in spending relative to a measure of past spending. The model analyzed here defines the utility of spending as a function of current spending divided by a power of the past spending peak. This ratio is at the heart of the model.

Instinctively, one can model shortfall aversion by adapting a form of the habit-dependent preferences model, pioneered by Sundaresan (1989). However, these preferences do not allow spending to equal, let alone fall below, the habit level. In the context of individuals or endowments this property is unreasonable. Cuts are more painful than comparable-sized gains, but should not be ruled out.

Any model of spending and investment must take a stand on the rate of time preference. Tobin (1974) asserts that “the trustees [of an endowment] are supposed to have a zero subjective rate of time preference.” This assumption is adopted here both because it may be substantively appropriate and because it keeps the model and its analysis clearer.

Prospect Theory considers choice among one-period lotteries and posits three deviations from the traditional theory of expected utility: (i) The existence of a reference level of wealth relative to which gains and losses are measured and valued; (ii) Higher marginal utility from losses than from gains; (iii) The weights of the different prospects are non-linear functions of their probabilities, in contrast to the usual expected value operator.

The model presented here has an internally-determined reference point, namely the highest spending rate to date, and higher marginal utility for a reduction of the rate of spending below the reference point than an increase above it. A spending increase above the reference point establishes a new, higher reference point which adversely affects the utility from future spending.

1.1 A Sketch of the Model

A specification of the classic Merton (1969; 1971) spending-investment problem is a convenient point of departure. It envisions a decision maker who continuously distributes wealth between current spending and saving (for future spending), which he allocates between a safe and a risky asset. There are no additional income sources (or claims on the wealth) and therefore no hedging motive to affect the spending-investment policy. His objective is to maximize the expected utility

of the open-ended spending path. A version of the problem with a rate of zero time preference and a single risky asset is to

$$\max_{c, \pi} \mathbb{E} \left[\int_0^\infty \frac{c_t^{1-\gamma}}{1-\gamma} dt \right] \quad (1.1)$$

over the consumption (c) and investment (π) policies, subject to a standard budget constraint.

This problem has a single preference-dependent parameter: the relative risk aversion γ . (The other parameters are market parameters of the joint return distribution of the safe and the risky asset.) The problem is solvable with a zero rate of time preference if $\gamma > 1$. (Without this further specification, prolonged periods of frugality followed by spending binges can achieve infinite welfare.)

According to the solution of the Merton model spending is proportional to wealth, as is the fraction of wealth invested in the risky asset. Explicitly, spending at time t equals to:

$$\hat{c}_t = m \hat{X}_t, \quad (1.2)$$

and the *Merton consumption fraction*

$$m = \left(1 - \frac{1}{\gamma}\right) \left(r + \frac{\mu^2}{2\gamma\sigma^2}\right) \quad (1.3)$$

where $\hat{X}_t$ is wealth at time t , r is the safe rate of return, μ is the excess return on the risky asset, and σ is the standard deviation of that return. The fraction of wealth in the risky asset is also constant, and given by the *Merton risky weight*

$$\pi = \frac{\mu}{\gamma\sigma^2}. \quad (1.4)$$

The novelty of the present model is that the instantaneous utility depends on spending relative to a function of a target which is past peak spending. Namely, the decision maker will

$$\max_{c, \pi} \mathbb{E} \left[\int_0^\infty \frac{(c_t/h_t^\alpha)^{1-\gamma}}{1-\gamma} dt \right] \quad (1.5)$$

subject to the usual budget constraint, which dictates that current savings will finance future spending, without other income sources or uses in the future. The variable h_t is the target relative to which current spending is enjoyed. It equals the maximum past spending rate

$$h_t = \max \left\{ \bar{h}, \sup_{s \in [0, t]} c_s \right\} \quad (1.6)$$

where $\bar{h}$ is the initial target, which represents the status quo inherited by the decision maker. (If no initial target is given, it is sufficient to consider it to be zero. An optimally-behaving decision maker will immediately increase it.)

The power α is the degree of shortfall aversion. It ranges between zero and one; the case $\alpha = 0$ corresponds to no shortfall aversion, which is the Merton model (1.1). With $\alpha > 0$, and with spending at the target level, the marginal utility of spending is higher for a slight cut in spending than for a slight increase in spending, which is how shortfall aversion is brought to bear. The ratio between the marginal utility of a spending cut and a spending increase at the target level (and

only there!) is $1 - \alpha$. In an experiments-based study Tversky and Kahneman (1992) suggest that a close analogue to α is about $1/2$.

At first glance, the modeling choice (1.5) and especially (1.6) seem to imply a commitment to an outlier in spending even if it took place in the very distant past. This concern is misplaced because the equilibrium spending path derived below is such that when current spending (the status quo) is below past peak spending – regardless how distant it is – then the equilibrium spending and investment policies are those of the Merton model. For past peak spending to affect spending and portfolio choices (which is when wealth is sufficiently high), the current spending is exactly at its historical peak. In fact, current spending is at the historical peak a substantial fraction of the time. (This fraction is equal approximately to α .)

1.2 A Sketch of the Solution

When choosing current spending and saving allocations, a forward looking planner takes into account the impact of current decisions on future benefits through a budget constraint (the standard impact) and also (and crucially) through the possibility that current spending sets a new spending peak and thereby changes the target h and affects the utility derived from future spending.

With preference and market parameters fixed, the decision maker's choice depends at each point on the ratio of wealth to the target. Figure 1 summarizes the main attributes of the closed-form solution, namely the optimal spending and investment rule. It depicts the evolution of the spending rule and of the portfolio weight of the risky asset as a function of the ratio of the wealth to target.

The Figure shows two regions and two boundary points. These points correspond to the gloom and bliss wealth to target ratios. The gloom ratio is $g = 1/m$, the inverse of the Merton consumption fraction (1.3). The bliss ratio is higher and given in (4.3) below. The target is constant except at the bliss point, where it increases to match (stochastic) increases in wealth if they take place. The inverse of the bliss point is the *lowest* spending rate as a fraction of wealth.

When the wealth to target ratio is at g ($= 1/m$) or lower, it is optimal to follow the Merton prescription for spending being proportional to wealth and the portfolio weight of risky asset being a constant, with both parameters equal to those derived by Merton. In this region wealth is so low relative to the established target that the optimal solution is unaffected by the discontinuity in the marginal utility at the point where spending is equal to the target. The inverse of the gloom point is the *highest* spending rate as a fraction of wealth.

When the wealth to target ratio is higher than g , and as long as it has not reached the bliss point, the optimal behavior entails constant dollar spending which is equal to the established target, and increasing the portfolio weight of the risky asset as the wealth to target ratio increases. This is the normal or target region.

The intuition underlying behavior in this region is straightforward: concern about establishing too high a target which will affect utility from future spending keeps spending from rising above the established target. The response to wealth changes is all in the portfolio weight of the risky asset. It increases with wealth because more wealth makes the target spending rate easier to preserve, thereby making the risky asset more attractive.

These two features of the spending and investment policy, which prevail approximately a fraction α of the time (in a sense to be made precise in Theorem 5.1 below), are in contrast with the Merton model. Spending is insensitive to changes in wealth in that region whereas the fraction invested in the risky asset increases if wealth increases.

Behavior at the bliss point is sensitive to the direction of wealth change. Deterioration of wealth takes the decision maker to the normal region and the behavior described above. In contrast, a wealth increase calls for an increase in spending (and thereby an increase in the target) so as to

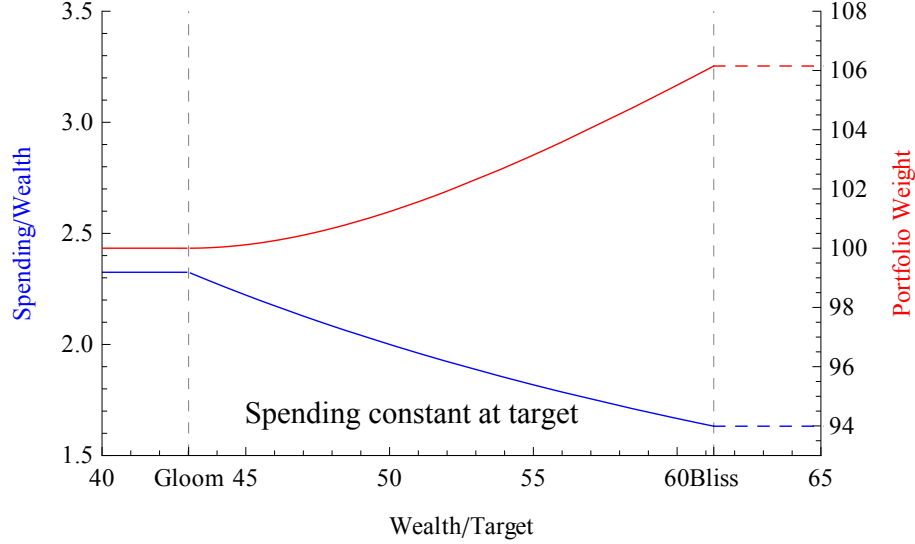


Figure 1: Ratio of optimal spending to wealth and portfolio weight of risky asset as functions of the wealth to target ratio, for market parameters the equity premium $\mu = 8\%$, the equity volatility $\sigma = 20\%$, the safe rate $r = .65\%$; and for endowment parameters risk aversion $\gamma = 2$ and shortfall aversion $\alpha = .5$.

maintain the wealth to target ratio. At this point the decision maker is so wealthy that the utility from immediate spending overwhelms concerns about the target being too high in the future. Thus, the optimal policy prevents any movement to a wealth to target ratio greater than the bliss ratio. At that point, the optimal portfolio weight of the risky asset is unaffected by wealth increase.

In summary, shortfall aversion induces two spending regimes: in good times (when wealth is between the gloom and bliss ratios) spending is constant at the target level, while in bad times (when wealth is below the gloom ratio) spending is proportional to wealth (in particular, it declines with wealth), so that spending is never higher, as a fraction of wealth, than at the gloom ratio. Vice versa, spending increases above the target when wealth reaches the bliss ratio, so that this ratio is never exceeded.

1.3 A Sketch of the Intuition

The classical condition that the marginal utility of optimal spending must match the marginal value of wealth or saving is essential. This condition is more complex in the good times region, where spending is at the target level and therefore the marginal utility of its increase is lower than that of its decrease. In this region the marginal value of wealth is greater than the marginal utility of spending increase and lower than the marginal utility of spending decrease. Therefore in that region a small addition to wealth is best used by adding to saving rather than to spending whereas it is best to subtract from saving rather than from spending a small loss of wealth.

At the boundary between the good times and bad times regions, spending is equal to the target

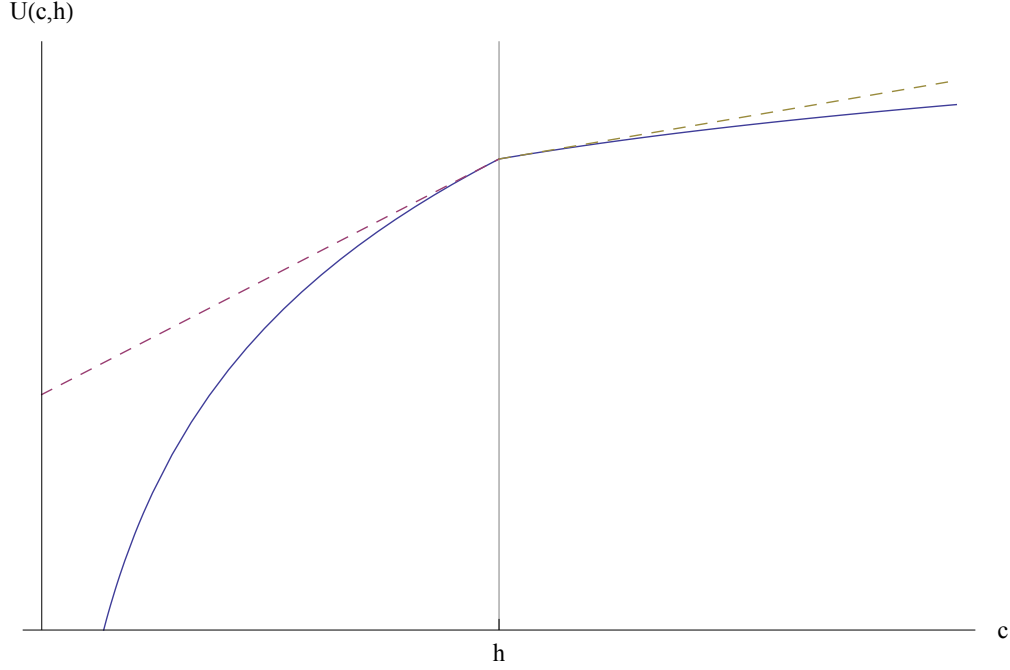


Figure 2: Utility from a spending rate c , when the target rate is h .

level but its increase and decrease have different implications for its marginal utilities. A decrease in spending means a shift to the bad times region where the marginal utilities of spending increase and decrease coincide, and are equal to the marginal value of wealth. Therefore in that region any change in wealth is allocated between savings and spending in a way that preserves the equality of their marginal utilities. A wealth increase from the boundary point between the good and bad times regimes means a shift to the good times regime.

At the higher end of the good times region, the marginal utility of an increase in saving is (infinitesimally) lower than the marginal utility of increase in spending (and in the target), and therefore increasing spending is the desired action, rendering the marginal utility of wealth and of spending equal.

1.4 Applications

Shortfall aversion of the beneficiary of a fund is key to the application. Shortfall aversion may be rooted in preferences, as suggested by the experimental evidence of Kahneman and Tversky, or associated with irreversible investments made in reliance on future spending of the fund's income.

A simple straightforward example is a trust fund to support a recipient into the indefinite future. It requires coherent spending-investment rules. The model studied here lays out the rules for a shortfall averse recipient in the special case that the endowment and its investment return are the only income sources.

Another simple example is of an endowment set up to support various causes which are autonomous and make plans and commitments for multiple years. At the margin, scaling back these plans is more costly than expanding the plans. Therefore the endowment is shortfall averse. The model offered here suggests how shortfall aversion affects its spending and investment policies.

Yet another possible application is to the payout and investment policy of trust funds set up by wealthy individuals for the benefits of generations of their off-springs. A simple trust has its

endowment as its sole source of spending and its spending plans are insensitive to the fortune (or lack thereof) of the beneficiaries.

Interpreting an infinite horizon model as a good approximation to a finite but long horizon model, one can adopt it to approximate the optimal spending and investing plan of a retiree in the early retirement years, if the retiree is shortfall averse. This paper's premise is that shortfall aversion is quite prevalent.

Beyond the applications suggested here, the paper proposes a way to model shortfall aversion and its initial analysis. Future work will examine extensions that entail discounting of the utility of future spending, discounting of the impact of the memory of past peak spending as well as stochastic future income.

The present model is normative rather than positive. It does not lend itself to aggregation because variation in wealth or history, let alone preferences, populate the model with individuals who cannot be summarized by a representative agent. Therefore, as it stands, the model is not designed to address questions in asset pricing. Simulations may overcome the challenge of heterogeneity; they are left to future work.

The next section offers a review of the literature and the one following it presents the model. The closed-form solution and its main properties are in Section 4; long-run properties of the solution are in Section 5. Section 6 entertains the possibility that a shortfall averse would apply the Merton spending-investment policy and concludes that for reasonable parameter values the loss would be the equivalent of 20% or more of initial wealth. Section 7 offers an illustration of the solution and its properties applied to the 1926-2012 market data. It suggests that low risk aversion and high shortfall aversion deliver high spending growth along a smooth path. A heuristic derivation of the solution is in Section 7, and Section 8 concludes. The Appendix offers the formal proof of the solution and its properties.

2 Literature Review

Merton (1969) is the classic model of dynamic spending-investment under uncertainty. Assuming a constant relative risk aversion utility of instantaneous spending, its time separability and asset prices following geometric Brownian motion, he argues that spending is a fixed proportion of wealth and that portfolio weights are independent of wealth. Within a similar analytic framework, Merton (1993) explicitly addresses the spending-investment problem of university endowments. He allows for additional income sources and spending needs whose evolution is governed by a Brownian motion. The solution is an adaptation of the earlier work to this more general case.

Merton (1973) takes the individual's spending-investment model to an aggregate level to derive a dynamic version of the CAPM. An implication is that aggregate consumption should be as volatile as aggregate wealth (including wealth invested in the safe asset). Stock market volatility and its weight in the aggregate wealth portfolio are in violation of this prediction, which is a paraphrase of the equity premium puzzle of Mehra and Prescott (1985).

To address the puzzle, Sundaresan (1989) – building on Ryder and Heal (1973) – considers a utility for instantaneous consumption which depends on the difference between current consumption and habit – a receding weighted average of past consumption. Constantinides (1990), Detemple and Zapatero (1991), Detemple and Zapatero (1992), and Campbell and Cochrane (1999) build on, and extend Sundaresan's original construct. (See also Detemple and Karatzas (2003) on internal habits and Menzly et al. (2004), Santos and Veronesi (2010) on external habits.) Common to these models is the prohibition on consumption to fall below the habit because marginal utility is infinity as soon as consumption is equal to the habit. Thus, the habit in these models is the *lowest possible*

consumption. This is in contrast with the target level of spending, which is the highest experienced consumption. The utility function proposed in the present model allows spending to fall below the target of past maximal spending, albeit reluctantly and painfully.

Abel (1990) offers a discrete time model of non-time separable utility function which contains a specification in which current utility depends on the ratio of current to previous period's consumption. (And, of course, next period's utility depends on the ratio of next to current period's consumption. Thus, current period's consumption affects directly the utility in two periods. Indirect effects could ripple further.) In Abel (1990) preferences are smooth and his goal is the study of asset pricing, especially the equity premium. In contrast, the present model posits an internally determined reference point at which preferences for spending are kinked.

Joining two power utility function delivers a utility which is continuous but not smooth at the attachment point in the present model. Andries (2012) employs a similar specification in her asset pricing model with loss aversion, but specifies a reference point that is log-linear in the value function, rather than past peak spending. Thus, her model entails a disutility from spending below *anticipated* level, in contrast with the present model in which disutility is derived from spending below *established* level.

Koszegi and Rabin (2006; 2007; 2008; 2009) develop a theory of reference-dependent preferences. The theory emphasizes consumption, tension between consumption and beliefs about consumption and possible tension between earlier and later beliefs about consumption. These tensions give rise to a gain-loss utility which is at the core of that work. Koszegi and Rabin do not address a Merton-like problem of consumption and investment. Pagel (2013*a*) builds on their work to study a dynamic model of life cycle consumption. Consumption comes from uncertain labor income and from saving at a safe rate. Pagel (2013*b*) further extends the model to allow for investment in a risky asset and derives the savings rate and risky asset portfolio weight when news regarding one's consumption affects one's well being. Assuming a representative agent with these expectations-based reference-dependent preferences, Pagel (2012) reports that the derived restrictions on the safe rate and the first two moments of the market returns and the consumption growth rate match the corresponding realized numbers.

Similar to the present paper, also Bilsen, Laeven and Nijman (2014) study the optimal spending and investment paths under loss aversion in spending, but model loss aversion differently and therefore derive a very different solution. In their approach, utility from consumption depends on the difference between the consumption rate and a reference rate, the dependence being sensitive to the sign of the difference. The reference rate itself is an exponentially weighted average of past consumption, as in the habit models. Bilsen, Laeven and Nijman (2014) argue that their model can be interpreted as a special case of the model analyzed by Kőszegi and Rabin (2006, 2007). The spending-investment model at the heart of the present paper also formalizes loss aversion in spending through reference-dependent preferences. The reference enters multiplicatively and is equal to past peak spending, not expectations about future spending.

The model presented in this paper is also a model of reference-dependent preferences which formalizes loss aversion. The reference, however, is past peak spending, not expectations about future spending. Thus, the reference is *actual* experience rather than expectations about future experience. This approach is reminiscent of realization utility as articulated in Barberis and Xiong (2012) and Ingersoll and Jin (2012), where investors derive utility from stocks they had purchased only when they sell them.

Deusenberry (1949) suggests a ratchet effect of consumption, i.e., that consumption is sensitive to its own path. Following an expansion, aggregate income decline is accompanied by a lesser decline in consumption, at the expense of a lower savings rate. Dybvig (1995, 1999) considers a close problem to the one studied here, namely the Merton problem (1.1) under the intolerance for

any decline in the standard of living, i.e., that spending cannot decrease over time. This condition in turn implies a spending rate always below the safe rate, as this is the maximum spending rate that can be sustained with certainty. The present model, in contrast, acknowledges that spending shortfall is painful, but does not rule it out. Accordingly, it implies realistic target spending rates well above the safe rate.

Shortfall aversion entails desirability of sustainability of the spending level but does not guarantee it. The degree of shortfall aversion α captures the effect of the past's peak spending on the utility from today's spending, but does not force the optimal solution to deliver a non-decreasing spending path. De Neve et al. (2014) hypothesize higher individual sensitivity to losses than to gains in economic growth. They report that various polls on measures of subjective well being are consistent with this hypothesis.

In contrast with the voluminous literature on spending and investment choice, not much has been written on the similar challenge facing endowments. Tobin (1974) is an early piece, titled "What Is Permanent Endowment Income?" It opens, "The trustees of an endowed institution are the guardians of the future against the claims of the present. Their task is to preserve equity among generations. The trustees of an endowed university like my own assume the institution to be immortal. They want to know, therefore, the rate of consumption from endowment which can be sustained indefinitely. Sustainable consumption is their conception of permanent endowment income. In formal terms, the trustees are supposed to have a zero subjective rate of time preference."

Gilbert and Hrdlicka (2012a) is an investigation into optimal policies of endowments, focused on the notion of fairness in a stochastic environment. Observing that efficiency considerations combined with attractive but risky rates of return may lead to a preference of future constituents at the expense of the current generation, the authors argue that an increasing preference for stochastic fairness reduces the allocation of endowment assets to risky assets, leading to lower payout rates approaching the risk-free rate. Gilbert and Hrdlicka (2012b) studies the appropriate university endowment objective function in the presence of stakeholders with diverse objectives and agency frictions. In contrast, the present paper adheres to the more traditional approach of maintaining tractability, at the expense of not incorporating these issues.

Brown et al. (2014) is an empirical study of the sensitivity of the spending policies of university endowments to changes in the endowments' wealth levels, the changes being mostly due to market returns. They note that following disappointing returns, university endowments tend to spend less than their stated policies. A structural model informed by this paper's analysis may help shed further light on the actual spending and investment choices of university endowments.

3 The Model

3.1 Preferences

A decision maker chooses a spending plan $(c_t)_{t \geq 0}$ to maximize the expected utility over an infinite horizon,

$$\mathbb{E} \left[\int_0^\infty U(c_t, h_t) dt \right]. \quad (3.1)$$

The utility function depends both on current spending c_t and on the *target* h_t , which is the maximum past spending level: if the endowment starts at time zero with a target of $\bar{h}$, at time t the target satisfies

$$h_t = \max \left\{ \bar{h}, \sup_{0 \leq s \leq t} c_s \right\}, \quad (3.2)$$

and the utility from spending c and target h has constant relative risk aversion $\gamma > 1$,

$$U(c, h) = \frac{(ch^{-\alpha})^{1-\gamma}}{1-\gamma}. \quad (3.3)$$

(With zero subjective time preference, $\gamma > 1$ is a necessary and sufficient condition for the problem to be well-posed. In its absence, expected utility can grow arbitrarily large by postponing spending.)

Two parameters summarize the preferences: relative risk aversion γ , and shortfall aversion α , which controls the target's effect on the instantaneous utility from current spending. Rewrite (3.3) (and implicitly (3.2)) as

$$U(c, h) = \begin{cases} \frac{(ch^{-\alpha})^{1-\gamma}}{1-\gamma} & c \leq h \\ \frac{c^{(1-\alpha)(1-\gamma)}}{1-\gamma} & c > h \end{cases}. \quad (3.4)$$

Note that the second expression follows from the first, since a choice of c above the target h instantly resets h to be equal to c . The risk aversion associated with spending reduction at h is γ . It is natural to consider the risk aversion associated with an increase in spending above h as $\gamma^* = 1 - (1 - \alpha)(1 - \gamma)$. It is the α -weighted average of one and γ ,

$$\gamma^* = \alpha + \gamma(1 - \alpha), \quad (3.5)$$

and it is always lower than γ since $\gamma > 1$.

Equation (3.4) implies, in particular, that at the target spending rate $c = h$, the marginal utility is $(1 - \alpha)h^{-\gamma^*}$ for an increase in spending, whereas it is $h^{-\gamma^*}$ for a decrease. Therefore, when spending is equal to the target, the marginal utility from cutting spending is higher than the marginal utility from increasing spending by a factor of $1/(1 - \alpha)$. This discontinuity in the marginal utility when spending equals the target is the key to the distance between the present model's recommendations and those of the earlier work. When $\alpha = 0$ the marginal utilities are equal and the model is identical to the traditional models, and when $\alpha = 1$ the ratio of the two marginal utilities is infinity, and increasing spending above the target has no advantages.

Discontinuity in marginal utility at a reference point is a key feature of Kahneman and Tversky's (1979, 1982) Prospect Theory, where it is called loss aversion. Much of that work of Kahneman and Tversky and the voluminous follow-up work is about aversion to losses of wealth, which presumably are eventually tied to losses in consumption. The present model is about shortfall in spending. The location of the reference point is always a challenging and delicate issue in models based on prospect theory. The choice here is that the reference point, i.e., the target, is endogenous and is equal to peak past spending.

Shortfall aversion is irrelevant for monotone spending plans. If spending c_t is increasing (and $\bar{h} = c_0$), then $h_t = c_t$ for all $t \geq 0$, and its utility is the same as in a standard model with risk aversion γ^* . Likewise, if spending is decreasing, then $h_t = \max\{\bar{h}, c_0\}$, and the utility is again the same as in a model with the same risk aversion γ . In particular, in a deterministic setting, where spending is increasing in view of positive interest rates (cf. (1.3) with $\mu = 0$), shortfall aversion is inconsequential. But it plays a central role in the present model because the endowment can finance its spending with a mix of safe and risky investments.

3.2 A Two-Asset Investment Opportunity Set

The financial market includes a safe asset with a fixed interest rate $r \geq 0$ and a risky asset the price of which follows a Brownian motion with excess expected returns μ , and volatility σ . The

Brownian motion is defined on a filtered probability space $(\Omega, \mathcal{F}, P)$; $(\mathcal{F}_t)_{t \geq 0}$ is the augmented natural filtration of W . The return on the risky asset satisfies

$$\frac{dS_t}{S_t} = (\mu + r)dt + \sigma dW_t. \quad (3.6)$$

At each time, the endowment chooses both the spending rate c_t and the fraction of remaining unspent wealth invested in the risky asset π_t . The self-financing condition requires that, with an initial capital $X_0 = x$, total wealth X_t satisfies the dynamics:

$$dX_t^{c, \pi} = (rX_t^{c, \pi} - c_t)dt + X_t^{c, \pi}\pi_t(\mu dt + \sigma dW_t), \quad (3.7)$$

The dynamics (3.7) motivates the definition of admissible spending-investment policies as follows:

Definition 3.1. An admissible strategy is a pair of adapted processes (c_t, π_t) , such that $\int_0^t c_s ds < \infty$ and $\int_0^t \pi_s^2 ds < \infty$ a.s. for all $t \geq 0$, and the corresponding wealth process $X^{c, \pi}$ in (3.7) satisfies $X_t^{c, \pi} \geq 0$ a.s. for all $t \geq 0$.

Denoting the class of admissible strategies by $\mathcal{A}$, the spending-investment problem defines the value function

$$V(x, \bar{h}) = \sup_{(c, \pi) \in \mathcal{A}} \mathbb{E}_{x, \bar{h}} \left[\int_0^\infty U(c_t, h_t) dt \right]. \quad (3.8)$$

where $\mathbb{E}_{x, \bar{h}}[\cdot]$ denotes for brevity the conditional expectation $\mathbb{E}[\cdot | X_0 = x, h_0 = \bar{h}]$.

3.3 The Optimal Policy With No Shortfall Aversion

With $\alpha = 0$, the model is the classical spending-investment problem considered by Merton, and its solution is the spending rate and risky asset weight (1.2) and (1.4), respectively. Both spending rate and the amount invested in the risky asset are fixed fractions of wealth.

Spending rate being proportional to wealth implies that their relative changes per unit of time are equal, i.e. $\frac{dc_t}{c_t} = \frac{dX_t}{X_t}$, and therefore the volatility of consumption equals the volatility of wealth

$$\frac{d\langle c \rangle_t}{c_t^2 dt} = \frac{d\langle X \rangle_t}{X_t^2 dt} = \frac{\mu^2}{\gamma^2 \sigma^2} \quad (3.9)$$

This implication of the benchmark model is counterfactual at the aggregate level, since consumption has a much lower volatility than asset prices. It is also problematic in applications, because highly stable spending is consistent only with very little risk, which in turn implies a consumption rate close to the safe rate (as $\gamma \uparrow \infty$ in (1.3), $c_t \approx rX_t$). In short, in the benchmark model, a stable consumption is a *small* fraction of wealth, nearly as small as the real interest rate.

As it stands, the benchmark model lacks the flexibility to allow for spending to be much smoother than wealth. But the reluctance to cut spending following wealth reductions appears universal as does avoidance of over-spending following unexpected enrichment. These features are modeled through shortfall aversion.

4 Optimal Spending and Investment

The Introduction already sketches the main attributes of the solution of the spending-investment problem. This section treats it formally, and discusses its main implications. It is qualitatively

different from the classical consumption-investment problem because of the discontinuity of the marginal utility function when spending is at the target level.

The expected return of the risky asset and its risk enter the bliss point through its *sharpe ratio* μ/σ , which is the risky asset's expected return scaled by its risk. On the whole, investment opportunities enter the bliss-gloom ratio through the parameter

$$\rho = \frac{2r}{(\mu/\sigma)^2}. \quad (4.1)$$

The average 1927-2011 US parameters are $r = 0.65\%$, $\mu = 8\%$ and $\sigma = 20\%$ per year. (Data are from Ibbotson.) Therefore a reasonable value for

$$\rho = 8\%. \quad (4.2)$$

The gloom point g (the wealth/target ratio below which spending cuts are necessary), as a fraction of the bliss point b (the wealth/target ratio at and above which spending should increase), depends on both the preference parameters – α and γ – and on the market parameters through ρ :

$$g = b \frac{(\alpha - 1)(\gamma - 1)(\rho + 1)r(\gamma\rho + 1)}{(\gamma - 1)(1 - \alpha)^{\rho+1} + (\gamma\rho + 1)(\alpha(\gamma - 1)(\rho + 1) - \gamma(\rho + 1) + 1)}. \quad (4.3)$$

For typical market values, equation (4.2) suggest that the gloom to bliss ratio is closely approximated by its limit for $\rho \downarrow 0$:

$$\frac{g}{b} \approx \frac{1 - \alpha}{1 - \alpha + \frac{\alpha}{\gamma^2} - \frac{1}{\gamma}(1 - \frac{1}{\gamma})(1 - \alpha) \log(1 - \alpha)} \quad (4.4)$$

This formula implies that the gloom point is a fraction of the bliss point that is rather insensitive to the exact value of market parameters, as long as the interest rate is small in comparison to the squared Sharpe ratio. This fraction depends both on risk aversion γ and on shortfall aversion α . For example, for γ close to 1, the ratio gloom/bliss approaches the constant $1 - \alpha$.

The paper's main result, a closed form solution of the endowment spending and portfolio policy, is next.

Theorem 4.1. *The optimal spending policy is:*

$$\hat{c}_t = \begin{cases} X_t/b & \text{if } h_t \leq X_t/b \\ h_t & \text{if } X_t/b \leq h_t \leq X_t/g \\ X_t/g & \text{if } h_t \geq X_t/g \end{cases} \quad (4.5)$$

The optimal weight of the risky asset is the Merton weight (1.4) when the wealth to target ratio is lower than the gloom point, $X_t/h_t \leq g$. Otherwise, i.e., when $X_t/h_t \geq g$ the weight of the risky asset is

$$\hat{\pi}_t = \frac{\rho(\gamma\rho + (\gamma - 1)z^{\rho+1} + 1)}{(\gamma\rho + 1)(\rho + (\gamma - 1)(\rho + 1)z) - (\gamma - 1)z^{\rho+1}} \frac{\mu}{\sigma^2} \quad (4.6)$$

where the variable z satisfies the equation:

$$\frac{(\gamma\rho + 1)(\rho + (\gamma - 1)(\rho + 1)z) - (\gamma - 1)z^{\rho+1}}{(\gamma - 1)(\rho + 1)rz(\gamma\rho + 1)} = \frac{x}{h}. \quad (4.7)$$

Equivalent expressions for $\hat{\pi}$ are

$$\hat{\pi}_t = \rho \left[\frac{h_t}{rX_t} \left(1 - \frac{1}{(1-\gamma)z} \right) - 1 \right] \frac{\mu}{\sigma^2} = \frac{2r}{\mu} \left[\frac{h_t}{rX_t} \left(1 - \frac{1}{(1-\gamma)z} \right) - 1 \right] \quad (4.8)$$

Remark. For ease of notation, this paper focuses on a single risky asset, but all the results extend immediately to several risky assets with a vector of expected returns μ and a covariance matrix Σ , replacing the term $(\mu/\sigma)^2$ in (1.3) and (4.1) with $\mu^\top \Sigma^{-1} \mu$, and the term μ/σ^2 in (4.6) with $\Sigma^{-1} \mu$. All the results in the rest of the paper remain valid with multiple risky assets with these substitutions.

Theorem 4.1 identifies the bliss and gloom points and states the optimal solution in the three regions they define. At the gloom point and for lower wealth to target ratios the solution is as in the benchmark case of Merton. Between the gloom and the bliss point spending is constant (i.e., insensitive to wealth changes) and equal to the spending rate at the gloom level. The portfolio weight of the risky asset is increasing in wealth in this region. Finally, wealth to target levels higher than that prescribed by the bliss ratio never materialize; if wealth increases at that point, spending increases accordingly, thereby establishing a higher target and keeping the endowment at the bliss point until deterioration in wealth takes the endowment back to the region between the bliss and the gloom point.

Implicitly, Theorem 4.1 covers also what happens initially. Namely, if the decision maker enters with a historical target $\bar{h}$ that is sufficiently high, the theorem covers the behavior at the initial instant and afterwards. If he enters with a very low target $\bar{h}$ or none, then initially he will spend a fraction $1/b$, of wealth, thereby resetting the initial target to the bliss level.

Unfortunately, with the exception of the benchmark case of $\alpha = 0$ it is impossible to express $\hat{\pi}$ directly in terms of x/h . Theorem 4.1 identifies the investment policy $\hat{\pi}$ in terms of the variable z , which is related to the wealth/target ratio x/h by equation (4.7). Equation (4.7) defines z in terms of primitives of the model. An alternative definition is

$$z = \frac{V_x(x, h)}{h^{-\gamma^*}}, \quad (4.9)$$

that is the marginal value of wealth $V_x(x, h)$, scaled by the marginal utility $h^{-\gamma^*}$ of target spending at the gloom point. The variable z is a decreasing function of wealth x , at the gloom point it is 1 and at the bliss point it is $1 - \alpha$. For wealth levels below the gloom point it can be arbitrarily large if wealth deteriorates. On the whole, along the optimal policy, the scaled marginal value of wealth z_t is a diffusion process with a reflecting boundary at $1 - \alpha$. (Compare subsection 8.2 below.)

Equation (4.6) implies that $\hat{\pi}_t$ decreases with z . At $z = 1$ it is equal to the weight of the risky asset in the Merton portfolio, (1.4) and as wealth increases (and z decreases) it increases. The limiting point $z = 0$ is interesting, corresponding to the bliss point for $\alpha = 1$. At that extreme point $\hat{\pi}_t = \mu/\sigma^2$ which is the Merton risky weight for the log utility function.

A comparison of the marginal value of wealth with that of the marginal utility of spending gives rise to the intuition underlying the optimal spending-saving allocation (4.5). Figure 3 illustrates this intuition which is made formal at (8.13) below. At wealth levels below the gloom ratio, the marginal utility of wealth and that of spending are equal and the first order condition reduces to that of Merton.

The picture changes as soon as the ratio of wealth to target is at the gloom point because at that point and at higher wealth levels the marginal utility of spending bifurcates into two marginal utilities: A higher one for cutting spending and a lower one for increasing it. Both marginal utilities of spending are constant in the target region because spending is constant in that region. Wealth

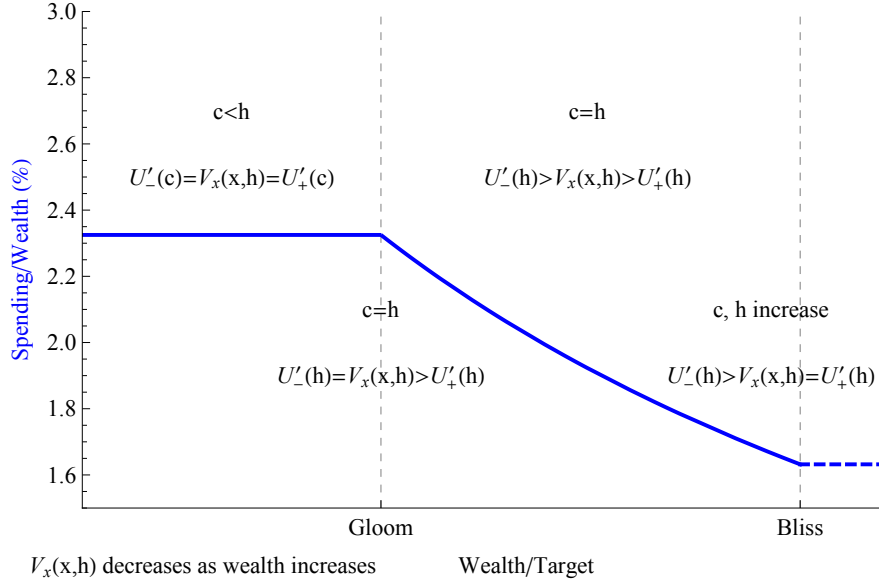


Figure 3: Ratio of optimal spending to wealth against the wealth to target ratio. At the left of gloom, spending is proportional to wealth (the Merton proportion). Between gloom and bliss, spending equals its past peak (the target region). Beyond bliss, spending increases instantly, reverting to the bliss point.

increases in that region with a movement from the gloom to the bliss ratio. At the gloom ratio the marginal value of wealth is equal to the marginal utility of cutting spending and above the marginal utility of increasing spending. An increase in wealth puts its marginal value between the marginal utilities of increasing and decreasing spending. Hence, at these points, an increase in wealth is associated with an increase in saving rather than in spending, whereas a decrease in wealth is associated with a reduction in saving rather than a reduction in spending.

As wealth increases, its marginal value decreases. At the bliss point the marginal value of wealth is so low that it is equal to the marginal utility of increasing spending. At this point an extra penny delivers higher overall utility if it is split between spending and saving, as to keep the wealth/target ratio at the same level.

The intuition underlying the behavior of the optimal weight of the risky asset, $\hat{\pi}$, parallels that of the intuition underlying the optimal spending behavior. At wealth levels below the gloom ratio, both variables follow the Merton formulas. In the target region, spending is as if the decision maker is poorer than he actually is, whereas the chosen exposure to risk is consistent with behavior as if the decision maker was wealthier than he actually is. Spending frugality allows higher exposure to risk. Another interpretation of investing a lower fraction of savings in the safe asset is through the observation that in the target region spending is locally insensitive to the investment outcome, thereby implicitly increasing the local tolerance for risk. Figure 4 displays the behavior of the portfolio weight of the risky asset as wealth/target varies, for different levels of shortfall aversion.

Next, Theorem 4.2 summarizes the sensitivities of the solution attributes to the parameters

Theorem 4.2. *The following properties hold:*

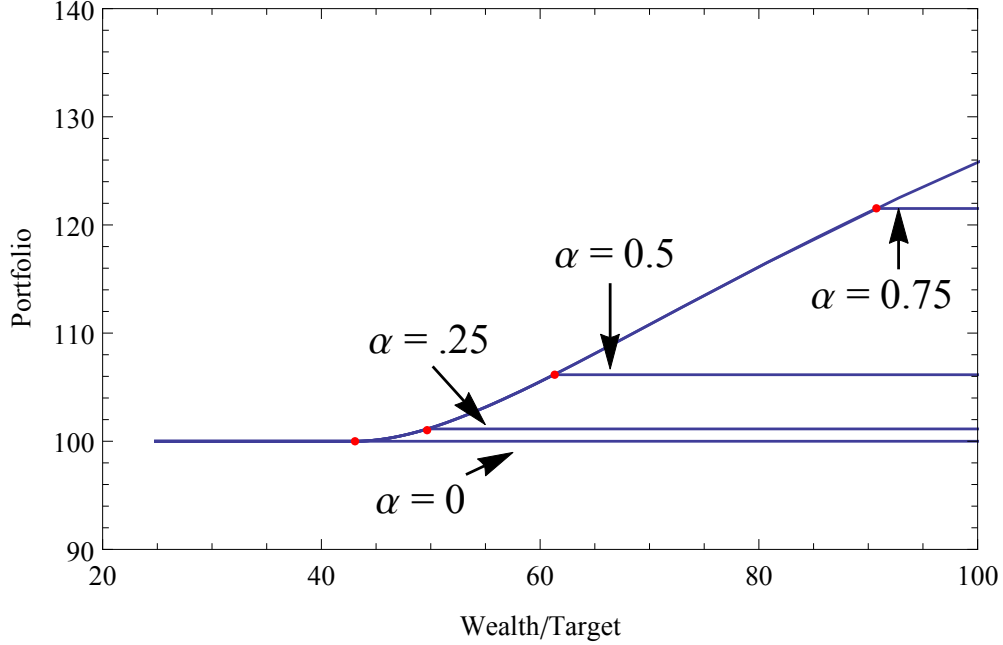


Figure 4: Portfolio weight (vertical axis) against wealth/target ratio, for different values of the shortfall aversion α .

- i) The gloom ratio is independent of the shortfall aversion α , and its inverse equals the Merton consumption rate, (1.3).
- ii) The bliss ratio (defined in (4.3)) increases as the shortfall aversion α increases.
- iii) Within the target region, the optimal portfolio weight of the risky asset $\hat{\pi}$ is independent of the shortfall aversion α .
- iv) At $\alpha = 0$ the model degenerates to the Merton model and $b = g$, i.e., the bliss and the gloom points coincide. At $\alpha = 1$ the bliss point is infinity, i.e., shortfall aversion is so strong that the solution calls for no spending increases at all.
- v) The gloom ratio and bliss ratio both approach the asymptotic value $1/r$ when the risk aversion γ approaches infinity. In particular, if $0 \leq \rho < 1$, the gloom ratio decreases for risk aversion close to one, reaches a minimum, and then increases toward the asymptotic value $1/r$.
 If $\rho - (1 - \alpha)^{\frac{\rho+1}{2}} < 0$, the bliss ratio decreases for risk aversion close to one, reaches a minimum, and then increases toward the asymptotic value $1/r$.
 If $\rho - (1 - \alpha)^{\frac{\rho+1}{2}} \geq 0$, the bliss ratio decreases to $1/r$ asymptotically.

Item (v) of the theorem is about the interaction of risk aversion and shortfall aversion. Its first assertion is based on the observation that at very high levels of risk aversion, the saving portfolio is almost entirely concentrated in the safe asset. Therefore the spending rate should be close to a fraction r of savings. Beyond the first assertion the item sketches the dependence of the gloom and bliss ratios on risk aversion.

5 Long-Run Properties of the Solution

Theorem 4.1 states the closed-form solution and Theorem 4.2 discusses comparative statics. This section offers further, long-run properties of the solution.

Theorem 5.1 below summarizes them. First, the Theorem states that the expected fraction of time that the spender-investor expects to be in the target region is approximately α . Then the Theorem considers an arbitrary starting point and states the expected time the spender-investor expects to spend until he reaches the gloom or the bliss point. The starting point is described in terms of z , the scaled marginal value of wealth defined in (4.9) or, equivalently, in (4.7). This scaled marginal value ranges from $1 - \alpha$ at the bliss point, to 1 at the gloom point, to any number above 1 to the left of the gloom point. Applying (4.7) one interprets the statement in terms of the wealth to target ratio, x/h .

Theorem 5.1.

- i) *The long-run average time spent in the target region is a fraction $1 - (1 - \alpha)^{1+\rho}$ of the total time. This fraction is approximately α because reasonable values of ρ are close to zero.*
- ii) *Starting from $z_0 \in [1 - \alpha, 1]$ corresponding to the initial wealth x and target h , the expected time to reach gloom is*

$$\mathbb{E}_{x,\bar{h}}[\tau_{gloom}] = \frac{\rho}{(\rho + 1)r} \left(\log(z_0) - \frac{(1 - \alpha)^{-\rho-1} (z_0^{\rho+1} - 1)}{\rho + 1} \right) \quad (5.1)$$

whence

$$\mathbb{E}_{x,\bar{h}}[\tau_{gloom}] = \frac{\rho}{r} \left(\frac{1 - z_0}{1 - \alpha} + \log z_0 \right) + O(\rho^2). \quad (5.2)$$

and in particular, starting from bliss ($z_0 = 1 - \alpha$),

$$\mathbb{E}_{x,\bar{h}}[\tau_{gloom}] = \frac{\rho}{r} \left(\frac{\alpha}{1 - \alpha} + \log(1 - \alpha) \right) + O(\rho^2) \quad (5.3)$$

- iii) *Starting from a point $z_0 \in [1 - \alpha, 1]$ in the target region, the expected time to reach bliss is*

$$\mathbb{E}_{x,\bar{h}}[\tau_{bliss}] = \frac{\rho}{r(\rho + 1)} \log \left(\frac{z_0}{1 - \alpha} \right). \quad (5.4)$$

In particular, starting from gloom ($z_0 = 1$), the expected time to reach bliss is

$$\mathbb{E}_{x,\bar{h}}[\tau_{bliss}] = \frac{\rho}{r(\rho + 1)} \log \left(\frac{1}{1 - \alpha} \right). \quad (5.5)$$

The average fraction of time in the target region being approximately α makes intuitive sense because at $\alpha = 0$ the model degenerates to the Merton model in which there is no target region; in contrast, shortfall aversion dominates at $\alpha = 1$ and the decision maker expects to spend almost all the time in the target region.

Both the time to reach gloom from bliss and the expected time to reach bliss from gloom increase with α , with both expected times being equal to zero for $\alpha = 0$ to both expected times being equal to infinity when $\alpha = 1$.

Next, Theorem 5.2 summarizes the long-run expected growth rate of savings.

Theorem 5.2. *The long-run return on the optimal portfolio, $\tilde{r} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (r + \mu \hat{\pi}_t) dt$, is*

$$\mathbb{E}[\tilde{r}] = r - \frac{\mu^2}{\sigma^2} (1 - \alpha)^{\rho+1} (\rho + 1) \int_{1-\alpha}^{\infty} \frac{q''(z)}{z^{\rho+1} q'(z)} dz \quad (5.6)$$

$$= r + (1 - \alpha)^{\rho+1} \frac{\mu^2}{\gamma \sigma^2} - \frac{\mu^2}{\sigma^2} (1 - \alpha)^{\rho+1} (\rho + 1) \int_{1-\alpha}^1 \frac{q''(z)}{z^{\rho+1} q'(z)} dz \quad (5.7)$$

where the function q is defined in Lemma A.4 below.

To understand this result, recall that in the Merton model the expected excess return is $\frac{\mu^2}{\gamma \sigma^2}$. (The optimal weight on the risky asset is $\frac{\mu}{\gamma \sigma^2}$.) In a similar vein, the second term in equation (5.7) represents the excess return of the Merton portfolio, weighted by the fraction of time that this portfolio is optimal – the time spent in the gloom region. The third term, which does not simplify further, represents the fraction of time spent in the target region (between gloom and bliss), times the corresponding excess return. This average return is higher than the one of the corresponding Merton portfolio because the exposure to the risky asset is higher in the target region.

Numerical calculations show that the long-run growth rate of savings increases with shortfall aversion. There are two sources of the extra growth rate: While the gloom regions of all spenders-investors are the same, the target region of the more shortfall averse contains that of the less shortfall averse, implying a more frugal spending policy. In addition, in the enlarged target region, the weight of the risky asset is higher.

A comparison across spender-investors with different shortfall aversions is instructive. Consider two types with identical initial wealth and levels of risk aversion γ , but with different levels of shortfall aversion, α_1 and α_2 . Equality of risk aversion guarantees that their gloom ratios are identical. The one with the higher shortfall aversion will have a higher bliss ratio. Assume that they both start with no spending history, so their spending rates move immediately to their respective bliss ratios. Therefore initially, the one with the higher shortfall aversion will be acting more frugally, spending a smaller fraction of his savings. Moreover, it is likely that before long the gap between their target levels will increase, further enhancing the relative savings of the spender-investor with the higher shortfall aversion. In addition, the more shortfall averse will put a higher fraction of his savings in the risky asset, and therefore they will grow at a higher average rate.

The higher bliss ratio of the more frugal implies that they will be saving more and their savings will grow at a higher average rate. Therefore not only will their savings be larger and grow faster, but eventually the more frugal will be spending more while spending smaller fractions of their wealth.

Continuing with the discussion of the two types, one can ask how long it takes for the spending of the more shortfall averse to catch up with (and subsequently exceed) that of the less shortfall averse. With $\alpha_1 < \alpha_2$ define the catch-up time to be $\tau_{\alpha_1, \alpha_2} = \inf\{t : h_t^1 \leq h_t^2\}$ where h^i is the target corresponding to α_i .

Theorem 5.3. *Let $\gamma_i^* = \alpha_i + (1 - \alpha_i)\gamma$ and $\psi = \frac{\gamma_1^* \gamma_2^* \sigma}{(\alpha_1 - \alpha_2)(1 - \gamma)\mu} \log \left[\frac{\gamma_1^* q(1 - \alpha_2)}{\gamma_2^* q(1 - \alpha_1)} \right] \geq 0$, where the function q is defined in Lemma A.4 below. Then*

$$\mathbb{P}(\tau_{\alpha_1, \alpha_2} \in dt) = \frac{\psi}{\sqrt{2\pi} t^3} \exp \left(- \frac{\left[\psi - \left(\frac{r\sigma}{\mu} + \frac{\mu}{2\sigma} \right) t \right]^2}{2t} \right) dt; \quad (5.8)$$

$$\mathbb{E}[\tau_{\alpha_1, \alpha_2}] = \frac{\psi}{\frac{r\sigma}{\mu} + \frac{\mu}{2\sigma}}. \quad (5.9)$$

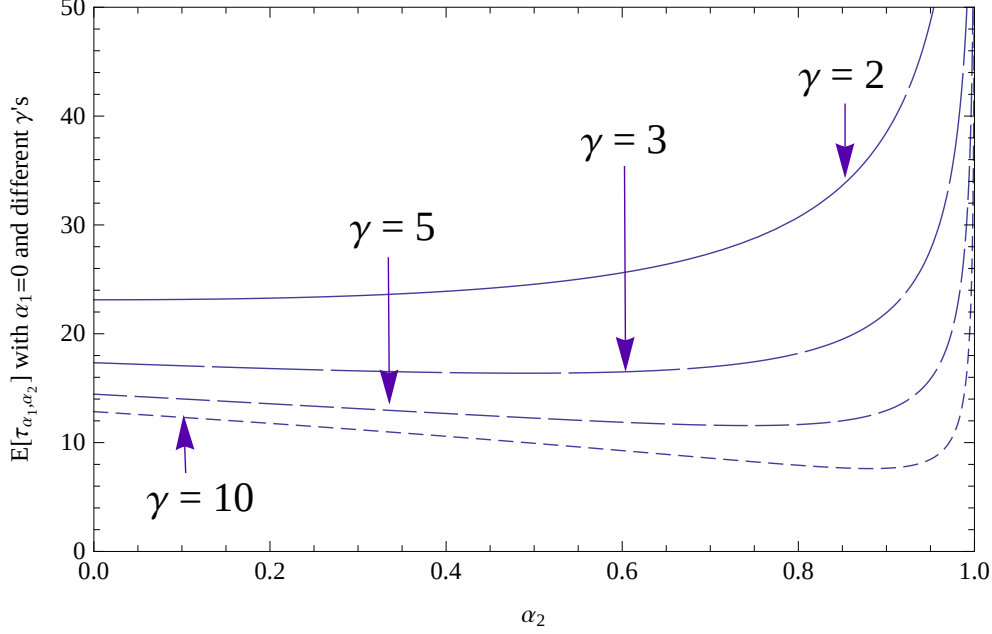


Figure 5: Expected time (vertical axis, in years) for the target spending of an agent with shortfall aversion α_2 (horizontal axis) to reach the target spending of an agent with no shortfall aversion, when both have risk aversion $\gamma = 2, 3, 5$ or 10 .

Figure 5 displays the average time it takes a shortfall-averse agent to spend as much as another agent with same risk aversion, but with no shortfall aversion. The first feature to note is that when the difference between the α 's is large then the expected time to equality of the targets increases with that difference. The reason is that when the difference in the α 's is higher, so is the difference in the initial spending and typically it takes a long time for the spending of the more frugal to catch up with that of the less frugal.

A second feature is the discontinuity at $\alpha_2 = \alpha_1$: trivially, $\tau_{\alpha, \alpha} = 0$, but $\lim_{\alpha_2 \downarrow \alpha_1} \mathbb{E}[\tau_{\alpha_1, \alpha_2}] > 0$. To understand the discontinuity, recall that the risky asset price may decline in which case both the more frugal and the less frugal expect to spend time also in the Merton region. Having the same risk aversion, in that region they spend and invest identically, regardless of a difference in their target spending, and therefore the wealth of the more frugal does not grow there relative to the wealth of the less frugal. The expected time spent in the Merton region is positive regardless how proximate the shortfall aversions of the more and the less frugal, and during that time no progress is made by the more frugal to increase his relative savings. A third feature is that for relatively low levels of difference in α 's, the expected time to convergence increases with that difference for low levels of risk aversion, and decreases with that difference for high levels of risk aversion. A fourth feature is that for a given level of risk aversion γ , the sensitivity of the expected time of spending catch-up is convex in the shortfall aversion α . Finally, a fifth feature is that for a given level of shortfall aversion α , the expected time of spending catch-up decreases with risk aversion γ .

6 The cost of ignoring shortfall aversion in the optimal policy

The Merton spending investment policy is simpler and better known than the optimal policy for the shortfall averse. A natural question is how much a shortfall averse loses by applying the Merton

α	h/x		
	$1/b$	$m(=1/g)$	$2m$
0.1	0.6	0.6	0.3
0.2	2.8	2.5	1.3
0.3	7.5	6.3	3.3
0.4	15.7	12.5	6.7
0.5	28.2	21.4	12.1
0.6	45.1	33.1	20.4
0.7	64.6	46.8	32.5
0.8	83.5	61.3	49.6
0.9	98.8	76.1	72.5

Table 1: The equivalent fractional wealth loss of the initial capital from applying the Merton policy instead of the optimal policy, for different levels of shortfall aversion α and for different initial spending targets h scaled by the Merton spending rate mx . Risk aversion is $\gamma = 2$, and market parameters are $\mu = 8\%$, $\sigma = 20\%$, $r = 0.65\%$. The parameter m ($= 1/g$) is in (1.3) and the relation between the bliss point b and the gloom point g is in (4.3).

policy rather than the optimal policy.

To address the question consider an individual with shortfall aversion α , risk aversion γ , initial wealth x , and initial target spending rate h . His expected utility over the open-ended time horizon under the optimal policy is $U^{OP}(x, \alpha, \gamma, h)$; it is $U^M(x, \alpha, \gamma, h)$ under the Merton policy. (The expected utility $U^{OP}(x, \alpha, \gamma, h)$ is an expedient notation for the value function defined in (3.8).) Trivially, $U^{OP}(x, \alpha, \gamma, h) \geq U^M(x, \alpha, \gamma, h)$ and $U^{OP}(x, 0, \gamma, h) = U^M(x, 0, \gamma, h)$.

Proposition 6.1 derives in closed form the *equivalent fractional wealth loss* L , defined by the solution of $U^{OP}(x(1 - L), \alpha, \gamma, h) = U^M(x, \alpha, \gamma, h)$. The solution L is the fraction of initial wealth effectively given up by a shortfall averse who applies the Merton spending-investment policy rather than the optimal one. It depends on α, γ , and h/x .

Proposition 6.1. *The equivalent fractional wealth loss*

$$L = 1 + \frac{h}{mx} q'(z) \quad (6.1)$$

where m is the Merton consumption fraction (1.3) and z solves the equation

$$q(z) - zq'(z) = \frac{(x/h)^{1-\gamma}}{1-\gamma} m^{-\gamma} \left(1 + \frac{\alpha}{1-\alpha} \left(\frac{h}{mx} \right)^{1-\gamma} \right). \quad (6.2)$$

Table 1 shows the equivalent fractional wealth loss for a range of values of shortfall aversion and for three values of initial target spending rate scaled by the Merton spending rate. An investor with risk aversion $\gamma = 2$ and shortfall aversion $\alpha = 0.5$ (close to the value suggested by Tversky and Kahneman (1992)), who starts with a target spending equal to the Merton spending, is indifferent between i) using the Merton policy with the initial wealth and ii) using the optimal policy with an initial wealth that is 21.4% lower. This is the cost of ignoring shortfall aversion when choosing the spending-investment strategy.

The larger shortfall aversion the bigger the loss from following the Merton policy rather than the optimal policy. On the other hand, the cost of following the Merton policy decreases as the initial target spending increases relative to the Merton rate: a higher initial target implies that the

optimal policy will coincide with the Merton strategy for a longer time, as wealth needs to reach a higher level before the target spending can be achieved, thereby exiting the gloom region. Yet, even for a target rate that is double the Merton spending (rightmost column in Table 1), the equivalent fractional loss is 12.1% of the initial wealth for shortfall aversion equaling 0.5, and higher for higher levels of shortfall aversion.

The numbers in Table 1 suggest that the shortfall averse who follow the Merton rather than the optimal policy make a substantial error.

7 Historical Performance with Market Returns (1926-2012)

To illustrate the analysis consider a shortfall averse decision maker who makes spending and investment choices annually, adhering to the policies derived in Theorem 4.1 and applying the market parameters which prevailed in 1926-2012 on average, namely $\mu = 8\%$ and $\sigma = 20\%$, $r = .65\%$. (Data are from the 2013 Ibbotson Yearbook. The equity premium μ is estimated as average of returns on large-capitalization equities, the safe rate r as the average real return on Treasury bills.)

Equation (4.5) describes the fraction of wealth allocated to spending at the beginning of each year and equation (4.6) describes the fraction of savings invested in the risky asset. The annual return on the safe and the risky asset are the US Treasury bill rate prevailing at the beginning of the year and the realized market return for that year, respectively.

Figure 6 summarizes the hypothetical wealth accumulation and spending under the Merton problem (the benchmark) and for a shortfall averse spender-investor. The preference parameters underlying Figure 6 are $\gamma = 2$ and α is either 0 or .5. Shortfall aversion manifests itself in a smoother spending path. In particular, along the spending path of the shortfall averse only in 25% of the years does spending fall below the target. In the Merton model ($\alpha = 0$) in 64% of the years spending is below its historical peak.

Table 2 summarizes five attributes of similar simulations done for various levels of risk aversion and shortfall aversion using the 1926-2012 data. The top two panels show average input variables – the average weight of the risky asset and the average spending rate. For each level of shortfall aversion, both decrease with risk aversion. In contrast, for a fixed level of risk aversion, both the average weight of the risky asset and the saving rate increase with shortfall aversion. (Saving rate is one minus the spending rate.) The monotonicity of the average portfolio return in α and in γ reflects the monotonicity of the weight of the risky asset in these parameters.

High shortfall aversion and low risk aversion are associated with high savings rate and high exposure to the risky asset. This combination leads to higher savings growth rate which in the long run leads to higher spending levels. The fraction of years in which shortfall is experienced decreases with α ; an extreme example is available for $\gamma = 2$ and $\alpha = 0.75$ where in 94% of the years spending does not fall below its historical peak. Moreover, conditioned on a shortfall, its magnitude decreases for higher levels of short aversion.

The example suggests an association between the combination of low risk aversion and high shortfall aversion and strong spending growth along a smooth path. In fact, the actual annual US consumption series has growth and volatility both approximately equal to 2% whereas the volatility of US stock returns has been about twice larger than that of the average return. The high stock market volatility relative to that of the consumption series is one of the facets of the Mehra and Prescott (1985) equity premium puzzle. A thus-far overlooked facet of the puzzle is that for a Merton representative agent the consumption series should be at its historical peak no more than 40% of the time (See the first column of Panel IV; unreported simulations confirm this observation for positive discount rates and lower levels of risk aversion.) This prediction is not borne by the

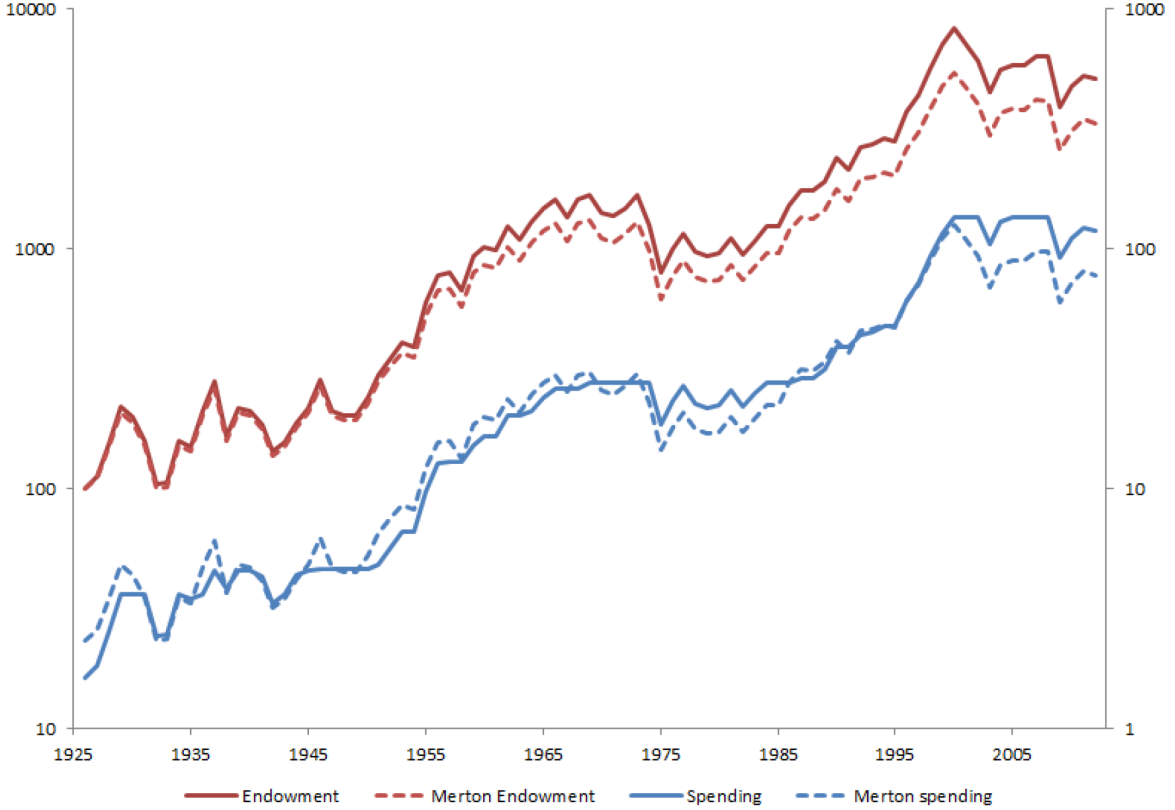


Figure 6: Savings (red) and annual spending rate (blue) with shortfall aversion $\alpha = 0.5$ (solid) versus the Merton benchmark of $\alpha = 0$ (dashed). Initial savings are at 100 and risk aversion $\gamma = 2$.

data. The annual consumption series has been at its historical peak in over 80% of the time, which is more consistent with a positive shortfall aversion.

8 Derivation of the Solution

This section sketches the heuristic control arguments underlying the derivation of the optimal spending and investment policies. The strategy is to derive the first order conditions governing the value function, then to impose boundary and smoothness constraints informed by economic intuition, obtain the tentative closed-form solution, and finally – in the Appendix – verify formally that the tentative solution is indeed optimal and therefore the solution.

The first subsection derives the relevant partial differential equation (PDE) and mentions some of its properties, recalling that the utility from spending depends also on the historical peak spending. The second subsection narrows the discussion to the utility function $U(c, h) = (c/h^\alpha)^{1-\gamma}/(1-\gamma)$ and reduces the PDE to a single ordinary differential equation (ODE) with a known solution.

Crucial in the second subsection is the observation that the utility function $U(c, h)$ retains the usual scaling property with respect to wealth, whereby doubling simultaneously wealth and spending (including the target) leads to a scaling of the utility by a constant. This property is key to arguing that the wealth to target ratio is an autonomous diffusion along the optimal path and deriving the analytic solution of that path.

I: Average portfolio weight of risky asset					II: Average spending rate				
α					α				
γ	0	0.25	0.5	0.75	γ	0	0.25	0.5	0.75
2	100	101	103	114	2	2.3	2.2	1.9	1.5
5	40	40	41	46	5	1.8	1.8	1.7	1.5
10	20	20	20	21	10	1.3	1.3	1.3	1.2

III: Average portfolio return					IV: Percentage of years at target spending					V: Average shortfall when below target				
α					α					α				
γ	0	0.25	0.5	0.75	γ	0	0.25	0.5	0.75	γ	0	0.25	0.5	0.75
2	8.5	8.5	8.7	9.4	2	36	50	75	94	2	26	21	16	7
5	3.8	3.8	3.9	4.2	5	32	46	62	80	5	16	14	12	7
10	2.3	2.3	2.3	2.4	10	23	26	32	45	10	20	19	17	15

Table 2: Summary statistics for spending and investments paths with the 1926-2012 realized returns for various values of risk and loss aversion. Market parameters are $\mu = 8\%$, $\sigma = 20\%$, $r = 0.65\%$.

8.1 Considerations for General Utility Functions

This subsection derives the Hamilton-Jacobi-Bellman PDEs ((8.4) and (8.5) below) for the value function $V(X_t, h_t)$ defined in (3.8). It then offers the solution for the portfolio weight of the risky asset in terms of the market parameters and derivatives of the value function (8.6). Using that solution it derives a non linear PDE (8.7) for the value function and for $\tilde{U}(y, h)$, the convex conjugate of the utility for spending U . Finally, it uses the convex conjugate (8.8) to turn the non linear PDE (8.7) into the linear PDE (8.11).

If one uses some (not necessarily optimal) policy (c, π) on $[0, t]$ and then switches to the optimal policy, the corresponding value is

$$J_t^{c, \pi} = \int_0^t U(c_s, h_s) ds + V(X_t, h_t), \quad (8.1)$$

where $h_t = \max\{\bar{h}, \sup_{s \leq t} c_t\}$ and the two terms in the right-hand side represent the spending and continuation values of the policy.

By Itô's formula, the time evolution of J is

$$dJ_t = L(X_t, \pi_t, c_t, h_t) dt + V_h(X_t, h_t) dh_t + V_x(X_t, h_t) X_t \pi_t \sigma dW_t, \quad (8.2)$$

where subscripts in x and h denote partial derivatives with respect to these variables, and

$$L(X_t, \pi_t, c_t, h_t) = U(c_t, h_t) + (X_t r - c_t + X_t \pi_t \mu) V_x(X_t, h_t) + \frac{V_{xx}(X_t, h_t)}{2} X_t^2 \pi_t^2 \sigma^2. \quad (8.3)$$

The martingale principle of optimal control (Davis and Varaiya, 1973) mandates that $J_t^{c, \pi}$ is a supermartingale for any policy, and a martingale for an optimal one. Since the process h_t increases only on the set $\{c_t = h_t\}$, the martingale principle requires that on the set $\{c_t < h_t\}$ the drift coefficient $L(X_t, \pi_t, c_t, h_t)$ is less than or equal to zero for any strategy, and equal to zero for the optimal strategy,

$$\sup_{c, \pi} L(x, \pi, c, h) = 0 \quad \text{for } c < h. \quad (8.4)$$

For $c = h$, the martingale principle requires that the coefficient of dh_t vanishes,

$$V_h(x, h) = 0 \quad \text{for } c = h. \quad (8.5)$$

The first step to solve the Hamilton-Jacobi Bellman equations (8.4) and (8.5) is to simplify the supremum in (8.4). Maximizing $L(x, \pi, c, h)$ with respect to π yields the first-order condition

$$\pi = -\frac{V_x}{xV_{xx}} \frac{\mu}{\sigma^2}, \quad (8.6)$$

which identifies the optimal portfolio in terms of the value function and its derivatives. In the classical Merton setting the indirect risk tolerance $-\frac{V_x}{xV_{xx}}$ is constant, and so is the optimal portfolio weight. Substituting (8.6) into (8.4), and defining $\tilde{U}(y, h) = \sup_{c \geq 0} [U(c, h) - cy]$ as the convex conjugate of U , equation (8.4) becomes:

$$\tilde{U}(V_x, h) + xrV_x(x, h) - \frac{V_x^2(x, h)}{2V_{xx}(x, h)} \frac{\mu^2}{\sigma^2} = 0. \quad (8.7)$$

This is a nonlinear differential equation for $V(x, h)$, which does not admit explicit solutions in general. To obtain a linear equation, consider the convex conjugate of the value function, defined by

$$\tilde{V}(y, h) = \sup_{x \geq 0} [V(x, h) - xy], \quad (8.8)$$

which depends on the current target h and on the variable y , which is interpreted as the current marginal utility of wealth. Indeed, the definition of $\tilde{V}$ implies the following relations with the value function V :

$$V_x(x, h) = y, \quad x = -\tilde{V}_y(y, h), \quad (8.9)$$

$$V_{xx}(x, h) = -1/\tilde{V}_{yy}(y, h), \quad V(x, h) = \tilde{V}(y, h) - y\tilde{V}_y(y, h). \quad (8.10)$$

Rewriting (8.7) in terms of $\tilde{V}$ and its derivatives, it becomes a linear differential equation:

$$\frac{\mu^2}{2\sigma^2} y^2 \tilde{V}_{yy} - ry\tilde{V}_y = -\tilde{U}(y, h). \quad (8.11)$$

8.2 Scaling Properties with Power Utility

The conjugate value function (8.8) depends both on the marginal utility of wealth y and on the target spending rate h . The next step is to exploit the scaling property of the utility function $U(c, h)$ to reduce this dependence to the single variable z , which is the marginal utility of wealth scaled by the marginal utility of spending at the gloom point.

This subsection applies the power utility (3.4) to the convex conjugate of the utility function U in (8.12), and, in (8.15) reduces the dimensionality of the conjugate value function. The latter and a change of variables lead to the simplified version of the Hamilton-Jacobi-Bellman equation as the second order ODE (8.17), with the available solution (8.20).

To proceed, recall (3.4) and calculate the conjugate utility $\tilde{U}$, which represents the maximum utility from spending, for marginal value of wealth y and target h . It is

$$\tilde{U}(y, h) = \sup_{c \geq 0} [U(c, h) - cy] = \begin{cases} \frac{h^{1-\gamma^*}}{1-\gamma} - hy & (1-\alpha) \leq yh^{\gamma^*} \leq 1, \\ \frac{\gamma}{1-\gamma} y^{1-1/\gamma} h^{-\alpha(1-\gamma)/\gamma} & yh^{\gamma^*} > 1, \end{cases} \quad (8.12)$$

while the $\tilde{U}(y, h)$ -maximizing spending rate c equals to

$$c = \begin{cases} h & (1 - \alpha) \leq yh^{\gamma^*} \leq 1, \\ y^{-1/\gamma} h^{-\alpha(1-\gamma)/\gamma} & yh^{\gamma^*} > 1. \end{cases} \quad (8.13)$$

In summary, the optimal spending and its utility have two different forms, depending on whether $z = yh^{\gamma^*}$, which represents the marginal utility or wealth relative to the marginal utility of the target's spending rate, lies in $[1 - \alpha, 1]$ or in $(1, \infty)$.

Further, the above power utility has the scaling property $U(\lambda c, \lambda h) = \lambda^{1-\gamma^*} U(c, h)$ for $\lambda > 0$. As in the Merton setting, this property is inherited by the value function, i.e. $V(\lambda x, \lambda h) = \lambda^{1-\gamma^*} V(x, h)$, and the conjugate function $\tilde{V}$ is also homogeneous, in that

$$\tilde{V}(\lambda^{-\gamma^*} y, \lambda h) = \sup_{x \geq 0} [V(\lambda x, \lambda h) - (\lambda x)(\lambda^{-\gamma^*} y)] = \sup_{x \geq 0} [\lambda^{1-\gamma^*} V(x, h) - \lambda^{1-\gamma^*} xy] = \lambda^{1-\gamma^*} \tilde{V}(y, h), \quad (8.14)$$

and with $\lambda = 1/h$, it follows that

$$\tilde{V}(y, h) = h^{1-\gamma^*} \tilde{V}(yh^{\gamma^*}, 1). \quad (8.15)$$

Denoting by $z = yh^{\gamma^*}$ the scaled marginal utility of wealth, and setting

$$q(z) = \tilde{V}(z, 1), \quad (8.16)$$

the above equation means that $\tilde{V}(y, h) = h^{1-\gamma^*} q(z)$ for some function q of the single variable z . Likewise, $\tilde{U}(y, h) = h^{1-\gamma^*} \tilde{U}(z, 1)$. Exploiting these properties, the PDE (8.11) reduces to the following ODE, defined piecewise

$$\frac{\mu^2}{2\sigma^2} z^2 q''(z) - rzq'(z) = -\tilde{U}(z, 1) = \begin{cases} z - \frac{1}{1-\gamma} & 1 - \alpha \leq z \leq 1, \\ -\frac{\gamma}{1-\gamma} z^{1-1/\gamma} & z > 1, \end{cases} \quad (8.17)$$

and in terms of q , the optimal policy is

$$c = \begin{cases} h & 1 - \alpha \leq z \leq 1, \\ hz^{-1/\gamma} & z > 1. \end{cases} \quad (8.18)$$

$$\pi = -\frac{zq''(z)}{q'(z)} \frac{\mu}{\sigma^2}. \quad (8.19)$$

The general solution to equation (8.17) is standard. For $r \neq 0$ it is

$$q(z) = \begin{cases} C_1 + C_2 z^{1+\rho} - \frac{z}{r} + \frac{2 \log z}{(2r + \frac{\mu^2}{\sigma^2})(1-\gamma)} & \text{if } 0 \leq z \leq 1, \\ C_3 + C_4 z^{1+\rho} + \frac{\gamma}{(1-\gamma)m} z^{1-1/\gamma} & \text{if } z > 1. \end{cases} \quad (8.20)$$

where C_1, C_2, C_3, C_4 are constants to be determined. The case $r = 0$ leads to a similar formula.

To identify the four constants in (8.20), four boundary conditions are needed. The first one is at $z = 1 - \alpha$, the optimality condition $V_h(x, h) = 0$ in (8.5). Recalling that $\tilde{V}_h(y, h) = -x$, and denoting by $\hat{y}(h)$ value of y such that $V(x, h) = \tilde{V}(\hat{y}, h) + x\hat{y}(h)$, it follows that

$$V_h(x, h) = \frac{\partial}{\partial h} (\tilde{V}(\hat{y}, h) + x\hat{y}(h)) = \tilde{V}_h(y, h) + (\tilde{V}_y(y, h) + x) \frac{d\hat{y}(h)}{dh} = \tilde{V}_h(y, h) \quad (8.21)$$

$$= \frac{d}{dh} (h^{1-\gamma^*} q(z)) = h^{-\gamma^*} [(1 - \gamma^*)q(z) + \gamma^* z q'(z)]. \quad (8.22)$$

where $z = yh^{\gamma^*}$. Thus, equation (8.5) in terms of $q(z)$ reduces to

$$(1 - \gamma^*)q(z) + \gamma^*zq'(z) = 0. \quad (8.23)$$

The definition of the convex conjugate of the value function $\tilde{V}$ implies (8.16), the definition of the function q , from which it naturally follows that the function $q(z)$ should be continuously differentiable, including at the point $z = 1$ where the two regions meet. The two requirements that follow are: value-matching ($q(1_-) = q(1_+)$), and smooth-pasting ($q'(1_-) = q'(1_+)$).

The fourth condition is obtained as z increases to infinity. Recall that z represents marginal utility relative to the target, and the natural condition is that marginal utility is infinite when wealth declines to zero. Since $q'(z) = \frac{\tilde{V}_y(y, h)}{h} = -\frac{x}{h}$ by (8.9), the fourth condition is

$$\lim_{z \rightarrow \infty} q'(z) = 0 \quad (8.24)$$

As a result, the boundary conditions (8.23) at the bliss point $1 - \alpha$ and (8.24) at bankruptcy, combined with the value-matching and smooth-pasting conditions at gloom, yield the four equations that identify the constants in the general form of the value function in (8.20).

9 Concluding Remarks

It appears far easier to accustom oneself to a higher standard of living than to a lower one. The model presented here focuses on this feature of preferences by scaling the utility of spending by a fractional power of past peak spending which is the target spending. When spending exceeds its own past peak, the target adjusts accordingly. Thus, the target is endogenous.

The analysis focuses on the ratio of wealth to target, which is a diffusion process with a reflecting barrier. At the lower levels of wealth to target the optimal spending rate is proportional to the wealth (and therefore spending is cut as wealth goes down in this region). Moreover, in this region the fraction of savings invested in the risky asset is fixed.

At the higher levels of wealth to target the spending rate is a constant, unaffected by changes in wealth. All changes in wealth translate immediately to commensurate changes in savings; moreover, increases (decreases) in savings translate to increases (decreases) in the fraction of savings invested in the risky asset. In this region the *payout rate* moves in the *opposite* direction of wealth, i.e., it increases as wealth decreases. At the reflecting barrier positive returns are partially consumed, raising the target spending rate.

An inspiration for the present model comes from the literature on Prospect Theory, which prominently features a reference point and an asymmetric reaction to changes up or down relative to the reference point, the down change being felt more strongly. Much of that literature is about wealth and changes thereof rather than consumption or spending, whereas the focus here is on spending. (Recall the title of Markowitz (1952) precursor of that literature, “The Utility of Wealth.”)

The location of the reference point is always a thorny issue, with many favoring the status quo as the reference point. In this paper the formal choice is to consider the historically highest spending rate as the reference point, which may seem very different from the status quo. However, in equilibrium, where it counts – in the target or normal region – the status quo is the historically highest spending rate.

The transition from the higher wealth to the target region to the lower one entails a reduction in payout rate which recalls the observations of Brown et al. (2014) regarding University endowments. They report

university endowments exhibit an asymmetric response to contemporaneous positive and negative financial shocks. Specifically, following positive shocks endowments tend to follow their own stated payout policies (e.g., pay out 5% of the past three- year average of endowment values). Whereas following contemporaneous negative shocks, many endowments actively deviate from their stated payout policies, actually reducing payout rates to a level below that implied by their standard smoothing rules.

Thus, Brown et al. (2014) observe two regions of behavior with payout policy being lower following negative financial shocks. Similarly, also the present model suggests that behavior should be different following negative financial shocks. However, the model suggests lower payout levels following a shock, but higher payout ratios. Moreover, the model suggests that in normal times payout levels should be constant (implying that payout rates should decrease with wealth) until they are increased due to a substantial wealth increase.

Another payout policy which recalls the results of the present model is that used by corporations to determine their dividends. Summarizing a seven-year, twenty-eight company study, Lintner (1956) writes,

With the possible exception of 2 companies... [c]onsideration of what dividends should be paid at any given time turned, first and foremost in every case, on the question of whether the existing rate of payment should be changed... We found no instance in which the question of how much should be paid in a given quarter or year was considered without regard to the existing rate as an optimum problem in terms of the interests of the company and/or its stockholders at the given time... ...serious consideration of the second question of just how large the change in dividend payments should be only after management had satisfied itself that a change in the existing rate would be positively desirable. Even then, the companies existing dividend rate continued to be a central bench mark for the problem in management's eyes... ... these elements of inertia and conservatism ... were strong enough that most managements sought to avoid making changes in their dividend rates that might have to be reversed within a year or so.

The general challenge is the reconciliation of a preference for smooth spending, and reluctance to cut back on spending with the desire to enjoy the higher returns associated with investments in risky assets. The model presented here has these properties; central and novel among these is the relative pain associated with a shortfall in spending.

A Appendix

This section contains the proofs of all the statements in the paper. The first subsection constructs the spending and investment policy explicitly, and proves that it is the unique optimal solution to the intertemporal utility maximization problem. In particular, Lemma A.1 uses a martingale argument to obtain an a priori upper bound on the utility of any spending-investment policy, while Lemma A.2 shows that this bound is achieved by the candidate optimal policy. Proposition A.3 and its auxiliary Lemmas A.4, A.5, A.8 link the optimal spending policy identified through the martingale argument with the solution of the ordinary differential equation in the paper, thereby providing the closed-form solution in Theorem 4.1.

The second subsection investigates the long-run properties of the optimal policy. Lemma A.9 identifies the invariant distribution of the marginal utility relative to the target, and hence the distribution of the spending rate relative to the target spending. Theorem A.11 proves that the risky asset weight is increasing in the target-wealth ratio.

A.1 The optimal spending and investment policy

The first task is to construct the optimal spending policy $\hat{c}_t$. For any constant $y \geq 0$, recall the dual of the utility function, $\tilde{U}(y, h) = \sup_{c \geq 0} (U(c, h) - cy)$, in (8.12):

$$\tilde{U}(y, h) = \begin{cases} -(1-\alpha)^{1/\gamma^*} \frac{y^{1-\frac{1}{\gamma^*}}}{1-\frac{1}{\gamma^*}} & y \leq (1-\alpha)h^{-\gamma^*} \\ \frac{h^{(1-\alpha)(1-\gamma)}}{1-\gamma} - hy & (1-\alpha)h^{-\gamma^*} < y \leq h^{-\gamma^*} \\ -\frac{y^{1-\frac{1}{\gamma}}}{1-\frac{1}{\gamma}} h^{\alpha(1-\frac{1}{\gamma})} & y > h^{-\gamma^*} \end{cases} \quad (\text{A.1})$$

while the $\tilde{U}(y, h)$ -maximizing spending rate $\hat{c}$ equals to

$$\hat{c} = \begin{cases} (y/(1-\alpha))^{-1/\gamma^*} & y \leq (1-\alpha)h^{-\gamma^*}, \\ h & (1-\alpha)h^{-\gamma^*} < y \leq h^{-\gamma^*}, \\ y^{-1/\gamma} h^{-\alpha(1-\gamma)/\gamma} & y > h^{-\gamma^*}. \end{cases} \quad (\text{A.2})$$

In particular, the expression $U(c, h) - cy$ attains its maximum for $c > h$ in the first case, for $c = h$ in the second case, and for $c < h$ in the third case. (Note that in our previous argument in subsection 8.2, the first case is merged to the second one, since once $c > h$, the target h will be updated and then $h = c$ as in the second case.) Note also that, when the first case holds, the value of $\tilde{U}$ does not depend on h .

Denote by M_t the *stochastic discount factor*

$$M_t := e^{-\left(r + \frac{\mu^2}{2\sigma^2}\right)t - \frac{\mu}{\sigma}W_t}. \quad (\text{A.3})$$

and recall that any spending-investment policy (c_t, π_t) starting from initial capital x satisfies the condition $E_{x, \bar{h}} [\int_0^\infty c_t M_t dt] \leq x$. The next result uses a martingale argument to find a family of upper bounds for the expected utility of any such spending plan.

Lemma A.1. *Any spending plan $(c_t)_{t \geq 0}$ such that $E_{x, \bar{h}} [\int_0^\infty c_t M_t dt] \leq x$ satisfies, for any $y > 0$:*

$$E_{x, \bar{h}} \left[\int_0^\infty \frac{(c_t h_t^{-\alpha})^{1-\gamma}}{1-\gamma} dt \right] \leq E_{x, \bar{h}} \left[\int_0^\infty \tilde{U}(y M_t, \hat{h}_t(y)) dt \right] + xy \quad (\text{A.4})$$

where $h_t = \bar{h} \vee \sup_{s \leq t} c_s$ and

$$\hat{h}_t(y) = \bar{h} \vee \left(y \inf_{s \leq t} M_s / (1-\alpha) \right)^{-1/\gamma^*}. \quad (\text{A.5})$$

Proof. First, note that for any $y > 0$ and any $(c_t)_{t \geq 0}$ such that $E_{x, \bar{h}} [\int_0^\infty c_t M_t dt] \leq x$,

$$E_{x, \bar{h}} \left[\int_0^\infty U(c_t, h_t) dt \right] = E_{x, \bar{h}} \left[\int_0^\infty (U(c_t, h_t) - y M_t c_t) dt \right] + y E_{x, \bar{h}} \left[\int_0^\infty c_t M_t dt \right] \quad (\text{A.6})$$

$$\leq E_{x, \bar{h}} \left[\int_0^\infty (U(c_t, h_t) - y M_t c_t) dt \right] + xy \quad (\text{A.7})$$

$$\leq E_{x, \bar{h}} \left[\int_0^\infty \tilde{U}(y M_t, h_t) dt \right] + xy \quad (\text{A.8})$$

The proof will be completed by showing that $\tilde{U}(yM_t, h_t) \leq \tilde{U}(yM_t, \hat{h}_t(y))$ for all $t \geq 0$.

Now, suppose that $h_t(\omega)$ strictly increases at t (henceforth ω is dropped for brevity), in that for any $\varepsilon > 0$ there is $u \in (t, t + \varepsilon)$ such that $h_u \geq c_u > h_t$. For any such u , the first case in (A.1) holds, and therefore

$$U(c_t, h_t) - yM_u c_t \leq \tilde{U}(yM_u, h_t) = \tilde{U}(yM_u, \hat{c}_u) \quad (\text{A.9})$$

where $\hat{c}_u > h_t$ satisfies the first-order condition $\hat{c}_u = (yM_u/(1 - \alpha))^{-1/\gamma^*}$. Passing to the limit as $u \downarrow t$, by the continuity of M and $\tilde{U}(\cdot, \cdot)$, it follows that

$$\tilde{U}(yM_t, h_t) = \tilde{U}(yM_t, \hat{c}_t) \quad (\text{A.10})$$

Now, in the path-wise sense, note that for any t , $h_t = \bar{h} \vee \sup_{s \in \mathcal{I}_t} h_s$, where $\mathcal{I}_t = \{s \leq t, h \text{ strictly increases at } s\}$. Then, $h_t = \bar{h} \vee \sup_{s \in \mathcal{I}_t} \hat{c}_s \leq \bar{h} \vee \sup_{s \leq t} \hat{c}_s = \hat{h}_t(y)$. Hence

$$\begin{aligned} U(c_t, h_t) - yM_t c_t &\leq \tilde{U}(yM_t, h_t) = \tilde{U}(yM_t, \bar{h} \vee \sup_{s \in \mathcal{I}_t} h_s) = \tilde{U}(yM_t, \bar{h} \vee \sup_{s \in \mathcal{I}_t} \hat{c}_s) \\ &\leq \tilde{U}(yM_t, \bar{h} \vee \sup_{s \leq t} \hat{c}_s) = \tilde{U}(yM_t, \hat{h}_t(y)) \end{aligned} \quad (\text{A.11})$$

□

In the above argument, the utility of a spending plan $(c_t)_{t \geq 0}$ achieves the upper bound if both the *first order condition* (A.12) and the *saturation condition* (A.13) hold:

$$\tilde{U}(yM_t, h_t) = U(c_t, h_t) - yM_t c_t, \quad (\text{A.12})$$

$$\mathbb{E}_{x, \bar{h}} \left[\int_0^\infty M_t c_t dt \right] = x. \quad (\text{A.13})$$

The next lemma shows that for any initial capital x there exists a constant $\hat{y} > 0$ and a spending plan $\hat{c}$ such that both conditions are satisfied, thereby proving the existence of an optimal policy.

Lemma A.2. *There exists a unique $\hat{y} > 0$ and a spending plan $\hat{c}_t(\hat{y})$ such that (A.4) holds as an equality. Hence, such plan is optimal.*

Proof. First, define the spending plan $\hat{c}_t(y)$ as in (8.13) for $\hat{h}_t(y) = \bar{h} \vee \left(\frac{\inf_{s \leq t} Y_s}{1 - \alpha} \right)^{-1/\gamma^*}$, with $Y_t = yM_t$, so that it satisfies the first-order condition. In other words, define

$$\hat{c}_t(y) = \hat{h}_t(y) F(Y_t), \quad (\text{A.14})$$

where

$$F(Y_t) = 1_{\{(1-\alpha)(\hat{h}_t(y))^{-\gamma^*} \leq Y_t \leq (\hat{h}_t(y))^{-\gamma^*}\}} + \left(Y_t (\hat{h}_t(y))^{\gamma^*} \right)^{-1/\gamma} 1_{\{Y_t > (\hat{h}_t(y))^{-\gamma^*}\}} \quad (\text{A.15})$$

$$= 1 + \left[\left(Y_t (\hat{h}_t(y))^{\gamma^*} \right)^{-1/\gamma} - 1 \right] 1_{\{Y_t > (\hat{h}_t(y))^{-\gamma^*}\}} \quad (\text{A.16})$$

$$= 1 + \left[\left(h^{-\gamma^*/\gamma} Y_t^{-1/\gamma} \wedge \left(\frac{\inf_{s \leq t} Y_s}{(1-\alpha)Y_t} \right)^{1/\gamma} \right) - 1 \right] 1_{\left\{ h^{-\gamma^*/\gamma} Y_t^{-1/\gamma} \wedge \left(\frac{\inf_{s \leq t} Y_s}{(1-\alpha)Y_t} \right)^{1/\gamma} < 1 \right\}}. \quad (\text{A.17})$$

This spending plan satisfies (A.12) by construction. To show that for some $y > 0$ it also satisfies (A.13), observe that when y increases, both $\hat{h}_t(y)$ and $F(Y_t)$ strictly decrease. Therefore

i) The following expectation is strictly decreasing with respect to y :

$$\mathbb{E}_{x,\bar{h}} \left[\int_0^\infty M_t \hat{c}_t(y) dt \right] = \mathbb{E}_{x,\bar{h}} \left[\int_0^\infty M_t \hat{h}_t(y) F(Y_t) dt \right].$$

ii) If $y \downarrow 0$, then $\hat{h}_t(y) \uparrow \infty$ and $F(Y_t) > 0$, implying

$$\mathbb{E}_{x,\bar{h}} \left[\int_0^\infty M_t \hat{c}_t(y) dt \right] \uparrow \infty.$$

iii) If $y \uparrow \infty$, then $\hat{h}_t(y) \downarrow h$ and $F(Y_t) \downarrow 0$, implying

$$\mathbb{E}_{x,\bar{h}} \left[\int_0^\infty M_t \hat{c}_t(y) dt \right] \downarrow 0.$$

Moreover, since $\mathbb{E}_{x,\bar{h}} [\int_0^\infty M_t \hat{c}_t(y) dt]$ is continuous in y , for any $x > 0$ there exists a unique $\hat{y}$ such that also the saturation condition holds. Since this policy $\hat{c}_t$ satisfies both the first order and saturation conditions, and in addition $\hat{h} \vee \sup_{s \leq t} \hat{c}_s(y) = \hat{h}_t(y)$, it follows that (A.4) holds as an equality, which means that $\hat{c}_t(y)$ is optimal. $\square$

The previous statements prove the existence of an optimal spending plan, and identify the value function as the minimum over y of the right-hand side of (A.4). The next step is to link this expression of the value function with $\tilde{V}(y, h)$, defined as the solution of the PDE (8.11) and its boundary conditions. By the homogeneity of the value function, for any $y \geq 0$, let $\tilde{V}(y, h) = h^{1-\gamma^*} q(z)$ with $z = h^{\gamma^*} y$, where q is defined in Lemma A.4.

Proposition A.3. *For any constant $y \geq 0$, the following holds for the $\hat{h}_t(y)$ defined in (A.5):*

$$\mathbb{E}_{x,\bar{h}} \int_0^\infty \tilde{U}(y M_t, \hat{h}_t(y)) dt = \tilde{V}(y, \bar{h}). \quad (\text{A.18})$$

The proof of Proposition A.3 hinges on a few technical Lemmas, which are proved afterwards.

Proof. Recall that for any $y \geq 0$, function $\tilde{V}(y, \bar{h})$ satisfies the partial differential equation:

$$\frac{(\mu/\sigma)^2}{2} y^2 \tilde{V}_{yy} - r y \tilde{V}_y + \tilde{U}(y, \bar{h}) = 0. \quad (\text{A.19})$$

Applying Itô's formula to $\tilde{V}(y M_t, \hat{h}_t(y))$, and using the equation above, it follows that

$$\begin{aligned} d \left(\tilde{V}(y M_t, \hat{h}_t(y)) \right) &= - \tilde{U}(y M_t, \hat{h}_t(y)) dt + \tilde{V}_h(y M_t, \hat{h}_t(y)) d\hat{h}_t(y) \\ &\quad - \frac{\mu}{\sigma} y M_t \tilde{V}_1(y M_t, \hat{h}_t(y)) dW_t. \end{aligned}$$

where, to avoid ambiguity, $\tilde{V}_1(\cdot, \cdot)$ denotes the partial derivative of $\tilde{V}(\cdot, \cdot)$ with respect to the first variable.

Define $\tau_n = \inf\{t \geq 0 \mid y M_t \geq n, \hat{h}_t(y) \geq [(1-\alpha)n]^{1/\gamma^*}\}$. By the form of $\hat{h}_t(y)$ in (A.5), it follows that $\inf_{s \leq t} (y M_s) \geq 1/n$ for all $t \leq \tau_n$. Now take any $T \in (0, \infty)$, and integrate the above from 0 to $T \wedge \tau_n$. Then, note that

i) The integral of the $d\hat{h}_t(y)$ term vanishes, because $\hat{h}_t(y)$ increases only if $\hat{c}_t(y) = \hat{h}_t(y)$, at which case $\tilde{V}_h = V_h = 0$ by the Neumann boundary condition in (8.5).

ii) The stochastic integral

$$\int_0^{T \wedge \tau_n} \frac{\mu}{\sigma} y M_t \tilde{V}_1(y M_t, \hat{h}_t(y)) dW_t = \int_0^{T \wedge \tau_n} \frac{\mu}{\sigma} y M_t \hat{h}_t(y) q' \left(y M_t (\hat{h}_t(y))^{\gamma^*} \right) dW_t. \quad (\text{A.20})$$

is a local martingale. In addition, by the continuity of the function q' (Lemma A.4 below) and the definition of τ_n , the expectation of the quadratic variation of this local martingale is finite, hence it is a martingale.

Then, taking expectations, it follows that:

$$\begin{aligned} \tilde{V}(y, \bar{h}) &= \mathbb{E}_{x, \bar{h}} \left[\int_0^{T \wedge \tau_n} \tilde{U}(y M_t, \hat{h}_t(y)) dt \right] + \mathbb{E}_{x, \bar{h}} \left[\tilde{V}(y M_{T \wedge \tau_n}, \hat{h}_{T \wedge \tau_n}(y)) \right] \\ &= \mathbb{E}_{x, \bar{h}} \left[\int_0^{T \wedge \tau_n} \tilde{U}(y M_t, \hat{h}_t(y)) dt \right] + \mathbb{E}_{x, \bar{h}} \left[\tilde{V}(y M_{\tau_n}, \hat{h}_{\tau_n}(y)) 1_{\{\tau_n \leq T\}} \right] \\ &\quad + \mathbb{E}_{x, \bar{h}} \left[\tilde{V}(y M_T, \hat{h}_T(y)) 1_{\{\tau_n > T\}} \right]. \end{aligned} \quad (\text{A.21})$$

Now consider the three expectations in (A.21) separately. For the first one, note that $\tilde{U} < 0$ since $\gamma > 1$, then as $n \rightarrow \infty$ the first term converges to

$$\mathbb{E}_{x, \bar{h}} \int_0^T \tilde{U}(y M_t, \hat{h}_t(y)) dt$$

by the monotone convergence theorem. The second term is bounded in absolute value by

$$E_{x, \bar{h}} \left[\left| \tilde{V}(y M_{\tau_n}, \hat{h}_{\tau_n}(y)) \right| 1_{\{\tau_n \leq T\}} \right] = E_{x, \bar{h}} \left[\left(\hat{h}_{\tau_n}(y) \right)^{1-\gamma^*} \left| q \left(y M_{\tau_n} (\hat{h}_{\tau_n}(y))^{\gamma^*} \right) \right| 1_{\{\tau_n \leq T\}} \right]. \quad (\text{A.22})$$

Since $\gamma^* > 1$, and by the definition of τ_n , it follows that $\left(\hat{h}_{\tau_n}(y) \right)^{1-\gamma^*} \leq \bar{h}^{1-\gamma^*}$, $y M_{\tau_n} \left(\hat{h}_{\tau_n}(y) \right)^{\gamma^*} \leq (1-\alpha)n^2$; and recall that (Lemma A.4 below)

$$q(z) = O(z^{1-1/\gamma}) \quad \text{as } z \rightarrow \infty, \quad (\text{A.23})$$

then

$$\left(\hat{h}_{\tau_n}(y) \right)^{1-\gamma^*} \left| q \left(y M_{\tau_n} (\hat{h}_{\tau_n}(y))^{\gamma^*} \right) \right| = O \left(n^{2(1-1/\gamma)} \right) \quad \text{for } n \rightarrow \infty. \quad (\text{A.24})$$

Moreover, by Chebyshev's inequality and (5.3.17) in Karatzas and Shreve (1991), there exists some constant C depending on m such that

$$\begin{aligned} \mathbb{E}_{x, \bar{h}} (1_{\{\tau_n \leq T\}}) &= \mathbb{P}_{x, \bar{h}} (\tau_n \leq T) \\ &\leq \mathbb{P}_{x, \bar{h}} \left(\left\{ \sup_{t \in [0, T]} y M_t \geq n \right\} \cup \left\{ \inf_{t \in [0, T]} y M_t \leq 1/n \right\} \right) \\ &\leq \mathbb{P}_{x, \bar{h}} \left(\sup_{t \in [0, T]} y M_t \geq n \right) + \mathbb{P}_{x, \bar{h}} \left(\inf_{t \in [0, T]} y M_t \leq 1/n \right) \\ &\leq \mathbb{P}_{x, \bar{h}} \left(\sup_{t \in [0, T]} y M_t \geq n \right) + \mathbb{P}_{x, \bar{h}} \left(\sup_{t \in [0, T]} y^{-1} M_t^{-1} \geq n \right) \\ &\leq n^{-2\kappa} E_{x, \bar{h}} \left[\sup_{t \in [0, T]} (y M_t)^{2\kappa} \right] + n^{-2\kappa} E_{x, \bar{h}} \left[\sup_{t \in [0, T]} (y M_t)^{-2\kappa} \right] \\ &= O \left(n^{-2\kappa} (1 + y^{2\kappa}) e^{CT} \right), \end{aligned} \quad (\text{A.25})$$

for any $\kappa \geq 1$. then (A.22) converges to 0 as $n \rightarrow \infty$, by (A.24) and (A.25). That is, the second expectation in (A.21) also converges to 0 as $n \rightarrow \infty$.

The third expectation in (A.21) converges as $n \rightarrow \infty$ to

$$E_{x,\bar{h}} \left[\tilde{V}(yM_T, \hat{h}_T(y)) \right].$$

Thus, passing to the limit as $n \rightarrow \infty$ in (A.21), it follows that

$$\tilde{V}(y, \bar{h}) = \mathbb{E}_{x,\bar{h}} \left[\int_0^T \tilde{U}(yM_t, \hat{h}_t(y)) dt \right] + E_{x,\bar{h}} \left[\tilde{V}(yM_T, \hat{h}_T(y)) \right]. \quad (\text{A.26})$$

The proof is completed, by observing that, as $T \rightarrow \infty$, the first expectation on the right hand side of (A.26) converges to

$$\mathbb{E}_{x,\bar{h}} \left[\int_0^\infty \tilde{U}(yM_t, \hat{h}_t(y)) dt \right]$$

by monotone convergence, and the second expectation converges to 0 by Lemma A.8 below. $\square$

The next lemma computes the explicit solution to the main ODE (8.17) and states its properties.

Lemma A.4. *The C^2 function $q : (1-\alpha, \infty) \rightarrow \mathbb{R}$ defined in the following cases, is convex and non-increasing on $(1-\alpha, \infty)$. (Recall that $m = \left(1 - \frac{1}{\gamma}\right) \left(r + \frac{\mu^2}{2\gamma\sigma^2}\right)$ denotes the Merton consumption fraction.)*

Case 1: $r \neq 0$.

$$q(z) = \begin{cases} C_1 + C_2 z^{1+\rho} - \frac{z}{r} + \frac{\rho \log z}{r(1-\gamma)(\rho+1)} & \text{if } 1-\alpha < z \leq 1, \\ C_3 + \frac{\gamma}{(1-\gamma)m} z^{1-1/\gamma} & \text{if } z > 1, \end{cases} \quad (\text{A.27})$$

where

$$\begin{aligned} C_2 &= \frac{1}{r(\rho+1)^2(\gamma\rho+1)} \\ C_1 &= -\frac{\gamma^*\rho+1}{1-\gamma}(1-\alpha)^\rho C_2 + \frac{1}{r(1-\gamma)} \left[1 - \frac{\rho}{\rho+1} \left(\frac{\gamma^*}{1-\gamma^*} + \log(1-\alpha) \right) \right] \\ C_3 &= C_1 + C_2 + \frac{1}{r} \left[\frac{\rho\gamma^3}{(\gamma-1)^2(\gamma\rho+1)} - 1 \right] \end{aligned}$$

Case 2: $r = 0$.

$$q(z) = \begin{cases} C_1 + C_2 z + \frac{2z \log z}{(\mu/\sigma)^2} + \frac{2 \log z}{(1-\gamma)(\mu/\sigma)^2} & \text{if } 1-\alpha < z \leq 1, \\ C_3 + \frac{\gamma}{(1-\gamma)m} z^{1-1/\gamma} & \text{if } z > 1, \end{cases} \quad (\text{A.28})$$

where

$$\begin{aligned} C_2 &= -\frac{2(\gamma+2)}{(\mu/\sigma)^2}, \\ C_1 &= \frac{2}{(1-\gamma)(\mu/\sigma)^2} \left[\alpha(\gamma-1) + 3 - \frac{1}{1-\gamma^*} - 2 \log(1-\alpha) \right], \\ C_3 &= \frac{2}{(1-\gamma)(\mu/\sigma)^2} \left[\alpha \left(\gamma - 1 - \frac{1}{1-\gamma^*} \right) - 2 \log(1-\alpha) \right]. \end{aligned}$$

Proof. For brevity, suppose $r \neq 0$, as the argument for $r = 0$ is analogous. To see that q is convex, consider its second derivative

$$q''(z) = \begin{cases} C_2(1+\rho)\rho z^{\rho-1} + \frac{\rho}{r(\gamma-1)(\rho+1)z^2} & \text{if } 1-\alpha < z \leq 1, \\ \frac{1}{\gamma m} z^{-1/\gamma-1} & \text{if } z > 1. \end{cases} \quad (\text{A.29})$$

Since C_2 is positive, $1 < 1+\rho$, and q is C^2 , it follows that q is convex on $(1-\alpha, \infty)$. Since q is convex, its derivative q' is nondecreasing, and since $q'(z) < 0$ for $z > 1$, it follows that q is nonincreasing for any $z > 1-\alpha$. $\square$

The next Lemma is used to prove Lemma A.8 below.

Lemma A.5. *Let $(B_t)_{t \geq 0}$ be a standard Brownian motion under the probability measure $\mathbb{P}$, and denote by $B_t^* = \sup_{0 \leq s \leq t} B_s$ its running maximum. Then, for any constants a, b, k with $2a + b \neq 0, k \geq 0$*

$$\begin{aligned} \mathbb{E} \left[e^{aB_T + bB_T^*} 1_{\{B_T^* > k\}} \right] &= \frac{2(a+b)}{2a+b} \exp \left\{ \frac{(a+b)^2}{2} T \right\} \Phi \left((a+b)\sqrt{T} - \frac{k}{\sqrt{T}} \right) \\ &\quad + \frac{2a}{2a+b} \exp \left\{ (2a+b)k + \frac{a^2}{2} T \right\} \Phi \left(-a\sqrt{T} - \frac{k}{\sqrt{T}} \right), \end{aligned}$$

and hence

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E} \left[e^{aB_T + bB_T^*} 1_{\{B_T^* > k\}} \right] = \begin{cases} \frac{(a+b)^2}{2} & \text{if } a+b > 0, 2a+b > 0, \\ \frac{a^2}{2} & \text{if } a < 0, 2a+b < 0, \\ 0 & \text{if } a+b \leq 0, a \geq 0. \end{cases}$$

where $\Phi(\cdot)$ is the standard normal distribution function.

Proof. Recall the joint probability density of (B_T, B_T^*) is

$$f_{B_T, B_T^*}(x, y) = \frac{2(2y-x)}{\sqrt{2\pi}t^{3/2}} e^{-\frac{(2y-x)^2}{2t}}, \text{ for } y \geq 0, x \leq y.$$

Hence the expectation is:

$$\begin{aligned} &\mathbb{E} \left[e^{aB_T + bB_T^*} 1_{\{B_T^* > k\}} \right] \\ &= \int_k^\infty \int_{-\infty}^y e^{ax+by} \frac{2(2y-x)}{\sqrt{2\pi}T^{3/2}} e^{-\frac{(2y-x)^2}{2T}} dx dy \\ &= \frac{2}{\sqrt{2\pi}T^{3/2}} \int_k^\infty e^{(2a+b)y + \frac{a^2}{2}T} \int_{-\infty}^y (2y-x) e^{-\frac{[x-(2y+aT)]^2}{2T}} dx dy \\ &= \frac{2}{\sqrt{2\pi}T^{3/2}} \int_k^\infty e^{(2a+b)y + \frac{a^2}{2}T} \left[T e^{-\frac{(y+aT)^2}{2T}} - \sqrt{2\pi}aT^{3/2} \Phi \left(-a\sqrt{T} - \frac{y}{\sqrt{T}} \right) \right] dy \\ &= \frac{2}{\sqrt{2\pi}T} \int_k^\infty e^{\frac{(a+b)^2}{2}T} e^{-\frac{[y-(a+b)T]^2}{2T}} dy - 2a \int_k^\infty e^{(2a+b)y + \frac{a^2}{2}T} \Phi \left(-a\sqrt{T} - \frac{y}{\sqrt{T}} \right) dy \\ &= 2e^{\frac{(a+b)^2}{2}T} \Phi \left((a+b)\sqrt{T} - \frac{k}{\sqrt{T}} \right) - 2a \int_k^\infty e^{(2a+b)y + \frac{a^2}{2}T} \Phi \left(-a\sqrt{T} - \frac{y}{\sqrt{T}} \right) dy, \end{aligned}$$

where the second term can be calculated by integration by parts

$$\begin{aligned}
& -2a \int_k^\infty e^{(2a+b)y + \frac{a^2 T}{2}} \Phi \left(-a\sqrt{T} - \frac{y}{\sqrt{T}} \right) dy \\
&= -\frac{2a}{2a+b} e^{\frac{a^2 T}{2}} \int_k^\infty \Phi \left(-a\sqrt{T} - \frac{y}{\sqrt{T}} \right) d e^{(2a+b)y} \\
&= -\frac{2a}{2a+b} e^{\frac{a^2 T}{2}} \left[-e^{(2a+b)k} \Phi \left(-a\sqrt{T} - \frac{k}{\sqrt{T}} \right) + \int_k^\infty \frac{1}{\sqrt{T}} e^{(2a+b)y} \phi \left(-a\sqrt{T} - \frac{y}{\sqrt{T}} \right) dy \right] \\
&= -\frac{2a}{2a+b} e^{\frac{a^2 T}{2}} \left[-e^{(2a+b)k} \Phi \left(-a\sqrt{T} - \frac{k}{\sqrt{T}} \right) + \frac{1}{\sqrt{2\pi T}} \int_k^\infty e^{(2a+b)y} e^{-\frac{(y+aT)^2}{2T}} dy \right] \\
&= \frac{2a}{2a+b} e^{\frac{a^2 T}{2}} \left[e^{(2a+b)k} \Phi \left(-a\sqrt{T} - \frac{k}{\sqrt{T}} \right) - e^{\frac{(2a+b)bT}{2}} \Phi \left((a+b)\sqrt{T} - \frac{k}{\sqrt{T}} \right) \right] \\
&= \frac{2a}{2a+b} e^{\frac{a^2 T}{2} + (2a+b)k} \Phi \left(-a\sqrt{T} - \frac{k}{\sqrt{T}} \right) - \frac{2a}{2a+b} e^{\frac{(a+b)^2 T}{2}} \Phi \left((a+b)\sqrt{T} - \frac{k}{\sqrt{T}} \right).
\end{aligned}$$

The lemma is followed. $\square$

Remark A.6. Note that when $k < 0$, then

$$\mathbb{E}_{x,\bar{h}} \left[e^{aB_T + bB_T^*} 1_{\{B_T^* > k\}} \right] = \mathbb{E}_{x,\bar{h}} \left[e^{aB_T + bB_T^*} \right],$$

and the corresponding limit $\lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_{x,\bar{h}} \left[e^{aB_T + bB_T^*} 1_{\{B_T^* > k\}} \right]$ is the same as in the above Lemma. That is, even if k is restricted to be nonnegative in Lemma A.5 for the expectation calculation, there is no restriction for k in the limit.

Corollary A.7. Let $B_t^{(\zeta)} = B_t + \zeta t$, where B is a standard Brownian motion under probability measure $\mathbb{P}$, $(B_t^{(\zeta)})^*$ be the supremum process of $B_t^{(\zeta)}$. Then for any constants a, b, k with $2a + b + 2\zeta \neq 0, k \geq 0$, the following expectation under $\mathbb{P}$ is:

$$\begin{aligned}
\mathbb{E} \left[e^{aB_T^{(\zeta)} + b(B_T^{(\zeta)})^*} 1_{\{(B_T^{(\zeta)})^* > k\}} \right] &= \frac{2(a+b+\zeta)}{2a+b+2\zeta} \exp \left\{ \frac{(a+b)(a+b+2\zeta)}{2} T \right\} \Phi \left((a+b+\zeta)\sqrt{T} - \frac{k}{\sqrt{T}} \right) \\
&+ \frac{2(a+\zeta)}{2a+b+2\zeta} \exp \left\{ (2a+b+2\zeta)k + \frac{a(a+2\zeta)}{2} T \right\} \Phi \left(-(a+\zeta)\sqrt{T} - \frac{k}{\sqrt{T}} \right),
\end{aligned}$$

and

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E} \left[e^{aB_T^{(\zeta)} + b(B_T^{(\zeta)})^*} 1_{\{(B_T^{(\zeta)})^* > k\}} \right] = \begin{cases} \frac{(a+b)(a+b+2\zeta)}{2} & \text{if } a+b+\zeta > 0, 2a+b+2\zeta > 0, \\ \frac{a(a+2\zeta)}{2} & \text{if } a+\zeta < 0, 2a+b+2\zeta < 0, \\ -\frac{\zeta^2}{2} & \text{if } a+b+\zeta \leq 0, a+\zeta \geq 0. \end{cases}$$

Proof. Find the probability measure $\mathbb{Q}$ by Girsanov's theorem so that $(B_t^{(\zeta)})_{t \geq 0}$ is a standard Brownian motion under $\mathbb{Q}$, and then calculate the expectation under $\mathbb{Q}$. $\square$

Lemma A.8. For any $y \geq 0$, $\lim_{T \rightarrow \infty} E_{x,\bar{h}} \left[\tilde{V}(yM_T, \hat{h}_T(y)) \right] = 0$.

Proof. By the form of function q , it follows that for z large

$$q(z) = O(z^{1-1/\gamma}). \quad (\text{A.30})$$

Therefore,

$$\begin{aligned}
E_{x,\bar{h}} \left[\tilde{V}(yM_T, \hat{h}_T(y)) \right] &= E_{x,\bar{h}} \left[(\hat{h}_T(y))^{1-\gamma^*} q \left(yM_T(\hat{h}_T(y))^{\gamma^*} \right) \right] \\
&= O \left(E_{x,\bar{h}} \left[(\hat{h}_T(y))^{1-\gamma^*} \left(yM_T(\hat{h}_T(y))^{\gamma^*} \right)^{1-1/\gamma} \right] \right) \\
&= O \left(E_{x,\bar{h}} \left[(\hat{h}_T(y))^{1-\gamma^*/\gamma} (M_T)^{1-1/\gamma} \right] \right). \tag{A.31}
\end{aligned}$$

Moreover,

$$\begin{aligned}
E_{x,\bar{h}} \left[(\hat{h}_T(y))^{1-\gamma^*/\gamma} (M_T)^{1-1/\gamma} \right] &= E_{x,\bar{h}} \left[\left(\bar{h} \vee \left(\frac{y \inf_{s \leq t} M_s}{1-\alpha} \right)^{-1/\gamma^*} \right)^{1-\gamma^*/\gamma} (M_T)^{1-1/\gamma} \right] \\
&\leq \bar{h}^{1-\gamma^*/\gamma} E_{x,\bar{h}} \left[(M_T)^{1-1/\gamma} \right] + E_{x,\bar{h}} \left[\left(\frac{y \inf_{s \leq t} M_s}{1-\alpha} \right)^{-1/\gamma^*+1/\gamma} (M_T)^{1-1/\gamma} 1_{\left\{ \left(\frac{y \inf_{s \leq t} M_s}{1-\alpha} \right)^{-1/\gamma^*} \geq \bar{h} \right\}} \right] \\
&= O \left(E_{x,\bar{h}} \left[(M_T)^{1-1/\gamma} \right] \right) + O \left(E_{x,\bar{h}} \left[\left(\inf_{s \leq t} (M_s) \right)^{-1/\gamma^*+1/\gamma} (M_T)^{1-1/\gamma} 1_{\left\{ \inf_{s \leq t} (M_s) \leq \frac{(1-\alpha)(\bar{h})^{-\gamma^*}}{y} \right\}} \right] \right). \tag{A.32}
\end{aligned}$$

The inequality above holds by observing that

$$\left(\bar{h} \vee \left(\frac{y \inf_{s \leq t} M_s}{1-\alpha} \right)^{-1/\gamma^*} \right)^{1-\gamma^*/\gamma} \leq \bar{h}^{1-\gamma^*/\gamma} + \left(\left(\frac{y \inf_{s \leq t} M_s}{1-\alpha} \right)^{-1/\gamma^*} \right)^{1-\gamma^*/\gamma} 1_{\left\{ \left(\frac{y \inf_{s \leq t} M_s}{1-\alpha} \right)^{-1/\gamma^*} \geq \bar{h} \right\}}. \tag{A.33}$$

To compute the two expectations in (A.32), note first that

$$M_T = \exp \left\{ - \left(r + \frac{\mu^2}{2\sigma^2} \right) T - \frac{\mu}{\sigma} W_T \right\} = \exp \left\{ - \frac{\mu}{\sigma} W_T^{(\zeta)} \right\},$$

where $W_T^{(\zeta)} = W_T + \zeta T$, $\zeta = \frac{r}{\mu/\sigma} + \frac{\mu/\sigma}{2}$. Therefore, the first expectation satisfies

$$\begin{aligned}
E_{x,\bar{h}} \left[(M_T)^{1-1/\gamma} \right] &= E_{x,\bar{h}} \left[\exp \left\{ - \left(1 - \frac{1}{\gamma} \right) \frac{\mu}{\sigma} W_T^{(\zeta)} \right\} \right] \\
&= \exp \left\{ - \left(1 - \frac{1}{\gamma} \right) \left(r + \frac{(\mu/\sigma)^2}{2\gamma} \right) T \right\} \quad \text{by Corollary A.7 with } b = k = 0,
\end{aligned}$$

Since $\gamma > 1$, it follows that

$$\lim_{T \rightarrow \infty} E_{x,\bar{h}} \left[(M_T)^{1-1/\gamma} \right] = 0 \tag{A.34}$$

On the other hand, for the second expectation term, apply Corollary A.7 again with $a = - \left(1 - \frac{1}{\gamma} \right) \frac{\mu}{\sigma}$, $b =$

$$\left(\frac{1}{\gamma^*} - \frac{1}{\gamma}\right) \frac{\mu}{\sigma}, k = \frac{\gamma^* \log h + \log y - \log(1-\alpha)}{\mu/\sigma}.$$

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T} \log \left\{ E_{x, \bar{h}} \left[\left(\inf_{s \leq t} M_s \right)^{-1/\gamma^* + 1/\gamma} (M_T)^{1-1/\gamma} 1_{\{\inf_{s \leq t} (M_s) \leq \frac{(1-\alpha)\bar{h} - \gamma^*}{y}\}} \right] \right\} \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \log E_{x, \bar{h}} \left[\exp \left\{ a W_T^{(\zeta)} + b \left(W_T^{(\zeta)} \right)^* \right\} 1_{\{(W_T^{(\zeta)})^* \geq k\}} \right] \\ &\leq \max \left\{ \frac{(a+b)(a+b+2\zeta)}{2}, \frac{a(a+2\zeta)}{2}, -\frac{\zeta^2}{2} \right\} \\ &= \max \left\{ -\left(1 - \frac{1}{\gamma^*}\right) \left(r + \frac{\mu^2}{2\gamma^* \sigma^2}\right), -\left(1 - \frac{1}{\gamma}\right) \left(r + \frac{\mu^2}{2\gamma \sigma^2}\right), -\frac{\zeta^2}{2} \right\} < 0. \end{aligned}$$

Hence,

$$\lim_{T \rightarrow \infty} E_{x, \bar{h}} \left[\left(\inf_{s \leq t} M_s \right)^{-1/\gamma^* + 1/\gamma} (M_T)^{1-1/\gamma} 1_{\{\inf_{s \leq t} (M_s) \leq \frac{(1-\alpha)\bar{h} - \gamma^*}{y}\}} \right] = 0. \quad (\text{A.35})$$

By (A.31), (A.32), (A.34) and (A.35), we can conclude that $E_{x, \bar{h}} \left[\tilde{V}(yM_T, \hat{h}_T(y)) \right]$ converges to 0 as $T \rightarrow \infty$. $\square$

Finally, to complete the proof of the main Theorem 4.1 it remains to verify that the optimal spending and investment policies have the feedback form in equations (4.5) and (4.6).

Proof of Theorem 4.1. Note that if $\tilde{V}(y, h)$ solves the PDE in (8.11), then $V(x, h) = \inf_{y>0} (\tilde{V}(y, h) + yx)$ solves the HJB equation (8.4) with the boundary condition (8.5). Thus, by Itô's formula implies that

$$\int_0^T U(c_t, h_t) dt \leq V(x, \bar{h}) - V(X_T, h_T) + \int_0^T V_x(X_t, h_t) X_t \pi_t \sigma dW_t \quad (\text{A.36})$$

Since $V_x(X_t, h_t)$ is square integrable, it follows that the last term is a martingale, and passing to the expectation:

$$E_{x, \bar{h}} \left[\int_0^T U(c_t, h_t) dt \right] \leq V(x, \bar{h}) - E_{x, \bar{h}} [V(X_T, h_T)] \quad (\text{A.37})$$

Finally,

$$V(X_T, h_T) = \inf_{y>0} (\tilde{V}(yM_T, h_T) + yM_T X_T) \geq \inf_{y>0} \tilde{V}(yM_T, h_T)$$

implies that for any $\varepsilon > 0$, there exists an $y^* > 0$ such that:

$$E_{x, \bar{h}} [V(X_T, h_T)] \geq E_{x, \bar{h}} \left[\inf_{y>0} \tilde{V}(yM_T, h_T) \right] \geq E_{x, \bar{h}} [\tilde{V}(y^* M_T, h_T)] - \varepsilon \quad (\text{A.38})$$

Take $T \rightarrow \infty, \varepsilon \rightarrow 0$, then

$$\lim_{T \rightarrow \infty} E_{x, \bar{h}} [V(X_T, h_T)] \geq \lim_{T \rightarrow \infty} E_{x, \bar{h}} [\tilde{V}(y^* M_T, h_T)] = 0 \quad (\text{A.39})$$

Thus, passing to the limit $T \uparrow \infty$ in (A.37) it follows that

$$E_{x, \bar{h}} \left[\int_0^\infty U(c_t, h_t) dt \right] \leq V(x, \bar{h}) \quad (\text{A.40})$$

On the other hand, since $V(x, h) = \inf_{y>0}(\tilde{V}(y, h) + xy)$ and $\tilde{V}(y, h) = h^{1-\gamma^*} q(yh^{\gamma^*})$, and for the $\hat{\pi}$ and $\hat{c}_t$ in (8.18) and (8.19) the inequality in (A.40) holds as an equality, it follows that these expressions define the optimal policies and $V(x, \bar{h})$ is the value function of the problem, completing the proof. $\square$

Proof of Theorem 4.2. i) By the definition of the gloom ratio in terms of the function q :

$$g = -q'(1) = 1/m \quad (\text{A.41})$$

where m is the Merton consumption ratio, and is independent of the shortfall aversion α .

ii) The bliss ratio can be expressed by the function q as $b = -q'(1 - \alpha)$. Then

$$\frac{db}{d\alpha} = q''(1 - \alpha) > 0 \quad (\text{A.42})$$

where the inequality holds because of the convexity of the function q .

- iii) Since z does not depend on α by (4.7), it is obvious that the optimal investment policy is also independent of the shortfall aversion α by (4.6).
- iv) When $\alpha = 0$, then (4.3) implies $g = b = 1/m$. That is, the model degenerates to the Merton model. When $\alpha = 1$, again by (4.3), it follows that the bliss point $b = \infty$.
- v) First, consider the dependence of the gloom ratio on risk aversion γ :

$$\frac{dg}{d\gamma} = -\frac{r}{m^2\gamma^3} \left[\gamma \left(1 - \frac{1}{\rho} \right) + \frac{2}{\rho} \right] \quad (\text{A.43})$$

If $0 < \rho < 1$, then the gloom ratio g decreases for risk aversion γ close to 1, reaching its minimum

$$g_{min} = \frac{4\rho}{r(\rho + 1)^2}$$

when $\gamma = \frac{2}{1-\rho}$, and then increases asymptotically to $1/r$.

If $\rho \geq 1$, then the gloom ratio g keeps decreasing with respect to risk aversion and approaches to $1/r$ asymptotically.

We then consider the dependence of the bliss ratio on risk aversion γ :

$$\frac{db}{d\gamma} = -\frac{\partial q'(1 - \alpha)}{\partial \gamma} \quad (\text{A.44})$$

$$= -\frac{\rho \left((1 - \alpha)^{\frac{\rho+1}{2}} (\gamma - 1) + (\gamma\rho + 1) \right) \left((1 - \alpha)^{\frac{\rho+1}{2}} (1 - \gamma) + (\gamma\rho + 1) \right)}{r(1 - \alpha)(1 - \gamma)^2(\rho + 1)(\gamma\rho + 1)^2} \quad (\text{A.45})$$

$$= -\frac{\rho \left((1 - \alpha)^{\frac{\rho+1}{2}} (\gamma - 1) + (\gamma\rho + 1) \right) \left[(\rho - (1 - \alpha)^{\frac{\rho+1}{2}})\gamma + ((1 - \alpha)^{\frac{\rho+1}{2}} + 1) \right]}{r(1 - \alpha)(1 - \gamma)^2(\rho + 1)(\gamma\rho + 1)^2} \quad (\text{A.46})$$

Each term in the last step of the above equation is obviously nonnegative, except the term $\left[(\rho - (1 - \alpha)^{\frac{\rho+1}{2}})\gamma + ((1 - \alpha)^{\frac{\rho+1}{2}} + 1) \right]$, which is positive when γ is around 1. That means, the bliss ratio decreases with respect to γ around 1.

Moreover, if $\rho - (1 - \alpha)^{\frac{\rho+1}{2}} < 0$, then the bliss ratio decreases to its minimum

$$b_{min} = \frac{(1 - \alpha)(\rho + 1)^2 - \left[(1 - \alpha)^{\frac{\rho+1}{2}} - \rho\right]^2}{r(1 - \alpha)(\rho + 1)^2}$$

when $\gamma = \frac{(1-\alpha)^{\frac{\rho+1}{2}} + 1}{(1-\alpha)^{\frac{\rho+1}{2}} - \rho}$, and then increases asymptotically to $1/r$.

If $\rho - (1 - \alpha)^{\frac{\rho+1}{2}} \geq 0$, the bliss ratio keeps decreasing and approaches $1/r$ asymptotically as γ is infinite. In such a case, the bliss ratio goes to infinity when γ is close to 1. $\square$

A.2 Long Run Properties of the Optimal Policy

As stated before, at time t , the optimal spending ratio $\hat{c}_t/X_t$ and investment fraction $\hat{\pi}_t$ only depends on h_t/X , the ratio of target spending over current wealth; moreover, the value functions only depends on the initial target/wealth h/x . That is, h_t/X_t plays an important role in our optimal problem. This section investigates the dynamics of process h/X , from which the dynamics of $\hat{c}, \hat{\pi}$ follow.

First define the following process $\{R_t\}_{t \in \mathbb{R}_+}, \{R_t^*\}_{t \in \mathbb{R}_+}, \{z_t\}_{t \in \mathbb{R}_+}$ via:

$$R_t = \frac{\hat{c}_t}{X_t}, \quad R_t^* = \frac{\hat{h}_t}{X_t}, \quad z_t = (\hat{h}_t)^{\gamma^*} V_x(X_t, \hat{h}_t). \quad (\text{A.47})$$

Then $\pi_t = -\frac{z_t q''(z_t)}{q'(z_t)} \frac{\mu}{\sigma^2}$, and the dynamics of the wealth process X is:

$$\frac{dX_t}{X_t} = \left(r - R_t - \frac{z_t q''(z_t)}{q'(z_t)} \frac{\mu^2}{\sigma^2} \right) dt - \frac{z_t q''(z_t)}{q'(z_t)} \frac{\mu}{\sigma} dW_t \quad (\text{A.48})$$

The dynamics of R^* is

$$\frac{dR_t^*}{R_t^*} = X_t d\left(\frac{1}{X_t}\right) + \frac{d\hat{h}_t}{\hat{h}_t} \quad (\text{A.49})$$

Therefore,

$$\frac{dR_t^*}{R_t^*} = \left(-r + R_t + \frac{z_t q''(z_t)}{q'(z_t)} \frac{\mu^2}{\sigma^2} + \frac{(z_t q''(z_t))^2 \mu^2}{(q'(z_t))^2 \sigma^2} \right) dt + \frac{z_t q''(z_t)}{q'(z_t)} \frac{\mu}{\sigma} dW_t + \frac{d\hat{h}_t}{\hat{h}_t} \quad (\text{A.50})$$

Now, consider process z . Recall that $R^* = -\frac{1}{q'(z)}$, or $z = p(-1/R^*)$, with p being the inverse function of q' . First, apply Itô's formula to $-1/R^*$:

$$d\left(-\frac{1}{R_t^*}\right) = \frac{1}{R_t^*} \left(-r + R_t + \frac{z_t q''(z_t)}{q'(z_t)} \frac{\mu^2}{\sigma^2} \right) dt + \frac{1}{R_t^*} \frac{z_t q''(z_t)}{q'(z_t)} \frac{\mu}{\sigma} dW_t + \frac{1}{R_t^*} \frac{d\hat{h}_t}{\hat{h}_t}$$

Then, apply Itô's formula again to $z = p(-1/R^*)$. Since p is the inverse function of q' ,

$$p' = \frac{1}{q''}, \quad p'' = -\frac{q'''}{(q'')^3}$$

Therefore, (note $q'(z_t) = -1/R_t^*$)

$$dz_t = \left(\frac{q'(z_t)}{q''(z_t)} (r - R_t) - \frac{\mu^2}{\sigma^2} z_t - \frac{\mu^2}{2\sigma^2} \frac{z_t^2 q'''(z_t)}{q''(z_t)} \right) dt - \frac{\mu}{\sigma} z_t dW_t + \gamma^* z_t \frac{d\hat{h}_t}{\hat{h}_t} \quad (\text{A.51})$$

Recall the HJB equation implies that (just take the derivative again, and use the fact that q is three times differentiable.)

$$\frac{\mu^2}{2\sigma^2} \cdot \frac{z^2 q'''(z)}{q''(z)} + \left(\frac{\mu^2}{\sigma^2} - r \right) z - \frac{r q'(z)}{q''(z)} = \begin{cases} \frac{1}{q''(z)} & \text{if } 1 - \alpha \leq z \leq 1, \\ \frac{z^{-1/\gamma}}{q''(z)} & \text{if } z > 1. \end{cases} \quad (\text{A.52})$$

We simplify the drift term of the process z in (A.51) in two cases:

1. When $1 - \alpha \leq z_t \leq 1$, i.e., between the bliss and gloom. Then $R_t = R_t^* = -1/q'(z_t)$. By (A.52), the drift term can be simplified as $-rz_t$.
2. When $z_t > 1$. Then $R_t = R_t^* z_t^{-1/\gamma} = -z_t^{-1/\gamma}/q'(z_t)$. Again, by (A.52), the drift term can be simplified as $-rz_t$.

Therefore, the drift is always $-rz_t$ for both cases.

The dynamics of the process z_t follows

$$z_t = z_0 - \int_0^t \frac{\mu}{\sigma} z_s dW_s - \int_0^t r z_s ds + \gamma^* \int_0^t z_s \frac{d\hat{h}_s}{\hat{h}_s}, \quad (\text{A.53})$$

Note that this stochastic differential equation of z does not depend on risk aversion γ before it hits the boundary $(1 - \alpha)$.

We are now ready to explore the recurrence and transience of process z . By straightforward calculation, we can find the *scale function* $\mathfrak{s}^z$ and *speed measure* m^z of z as follows:

$$\mathfrak{s}^z(z) = \frac{1 - \alpha}{1 + \rho} \left[\left(\frac{z}{1 - \alpha} \right)^{1 + \rho} - 1 \right] \quad (\text{A.54})$$

$$m^z(dz) = \frac{2\rho(1 - \alpha)^\rho}{r} z^{-\rho - 2}. \quad (\text{A.55})$$

Lemma A.9. *The process z is positively recurrent, with invariant distribution:*

$$\nu(dz) = 1_{\{1 - \alpha \leq z < \infty\}} \frac{1 + \rho}{1 - \alpha} \left(\frac{z}{1 - \alpha} \right)^{-\rho - 2}. \quad (\text{A.56})$$

Proof. The recurrence of z is a direct result of Karatzas and Shreve (1991) (Proposition 5.5.22). Moreover, straightforward calculation shows that:

$$\int_{1 - \alpha}^{\infty} m(dz) < \infty.$$

According to Borodin and Salminen (2002)(II.2.12), process z is positively recurrent, and its invariant distribution is the normalized speed measure:

$$\nu(dz) = \frac{m(dz)}{m([1 - \alpha, \infty))} = 1_{\{1 - \alpha \leq z < \infty\}} \frac{1 + \rho}{1 - \alpha} \left(\frac{z}{1 - \alpha} \right)^{-\rho - 2}. \quad (\text{A.57})$$

□

Proof of Theorem 5.1. i) Recall that when the spending is in the target region, the process z is between 0 and 1. By ergodic theorem and the form of $\nu(\cdot)$ in (A.57), a straightforward calculation gives that

$$\lim_{T \rightarrow \infty} \frac{\int_0^T 1_{\{1-\alpha < z_s < 1\}} ds}{T} = \int_{1-\alpha}^1 \nu(z) dz = 1 - (1 - \alpha)^{1+\rho}.$$

ii) Let $g(z)$ be the solution to the following ordinary differential equation with boundary conditions:

$$\begin{aligned} \frac{\mu^2}{2\sigma^2} z^2 g''(z) - r z g'(z) &= -1, \text{ for } z \in (1 - \alpha, 1) \\ g(1) &= 0, \quad g'(1 - \alpha) = 0. \end{aligned}$$

Then $g(z)$ can be solved as:

$$g(z) = \frac{\rho}{(\rho + 1)r} \left(\log(z) - \frac{(1 - \alpha)^{-\rho-1} (z^{\rho+1} - 1)}{\rho + 1} \right) \quad (\text{A.58})$$

Apply Itô's formula to $g(z_t)$, and integrate from 0 to τ_{gloom} , then by the dynamic equation of z_t in (A.53):

$$g(z_{\tau_{gloom}}) - g(z_0) = -\tau_{gloom} - \frac{\mu}{\sigma} \int_0^{\tau_{gloom}} z_s g'(z_s) dW_s + \gamma^* \int_0^{\tau_{gloom}} \frac{z_s g'(z_s)}{\hat{h}_s} d\hat{h}_s. \quad (\text{A.59})$$

Taking expectations, and note that

- (1) $g(z_{\tau_{gloom}}) = g(1) = 0$.
- (2) the stochastic integral is square-integrable, therefore it is a martingale, and has zero mean.
- (3) The nondecreasing process $\hat{h}_t$ only increases on $\{z_t = 1 - \alpha\}$, but $g'(1 - \alpha) = 0$, implying $\int_0^{\tau_{gloom}} \frac{z_s g'(z_s)}{\hat{h}_s} d\hat{h}_s = 0$

Therefore, $\mathbb{E}\tau_{gloom} = g(z_0)$ as desired.

iii) Note that the first time to reach bliss can be defined as:

$$\tau_{bliss} = \inf\{t \in \mathbb{R}_+ \mid z_t = 1 - \alpha\}. \quad (\text{A.60})$$

and recall that the process z reflects only at the boundary $1 - \alpha$. Hence before time τ_{bliss} , the process z behaves like a geometric Brownian motion:

$$z_t = z_0 - \int_0^t \frac{\mu}{\sigma} z_s dW_s - \int_0^t r z_s ds.$$

which has the explicit solution $z_t = z_0 \exp\{-\frac{\mu}{\sigma} W_t - (r + \frac{\mu^2}{2\sigma^2})t\}$. Then

$$\{z_t \leq 1 - \alpha\} = \left\{ -\frac{\mu}{\sigma} W_t - \left(r + \frac{\mu^2}{2\sigma^2}\right)t \leq -\ln\left(\frac{z_0}{1 - \alpha}\right) \right\} \quad (\text{A.61})$$

Let process B be a 1-dimensional Brownian motion under the probability measure $\mathbb{P}$. Then by applying equation (9.3) in Rogers and Williams (n.d.), with $a = -\infty$, $b = \frac{1}{\sqrt{(\mu/\sigma)^2}} \ln \left(\frac{z_0}{1-\alpha} \right)$, $c = \frac{r}{\sqrt{(\mu/\sigma)^2}} + \frac{\sqrt{(\mu/\sigma)^2}}{2}$, $\beta = \sqrt{c^2 + 2\lambda} - c$, we have that for any $\lambda > 0$:

$$\mathbb{E}[e^{-\lambda\tau_{bliss}}] = e^{-b\beta} \quad (\text{A.62})$$

Then as desired,

$$\mathbb{E}[\tau_{bliss}] = -\frac{d\mathbb{E}[e^{-\lambda\tau_{bliss}}]}{d\lambda} \Big|_{\lambda=0} = \frac{b}{c} = \frac{\rho}{r(\rho+1)} \log \left(\frac{z_0}{1-\alpha} \right)$$

□

Lemma A.10.

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}[\log X_T] &= r - r(1-\alpha)^{\rho+1} \left(\frac{\gamma\rho - \rho - 1}{\gamma\rho} \right) \\ &\quad + (1+\rho)(1-\alpha)^{\rho+1} \int_{1-\alpha}^1 \frac{1}{z^{\rho+1}} \left\{ \frac{1}{q'(z)} - \frac{zq''(z)}{q'(z)} \frac{\mu^2}{\sigma^2} \left(1 + \frac{zq''(z)}{2q'(z)} \right) \right\} dz \end{aligned} \quad (\text{A.63})$$

Proof. Apply Itô's formula to $\log X_T$, together with the dynamic of the wealth process under our optimal strategy in equation (A.48):

$$\log X_T = \int_0^T r - R_t - \frac{z_t q''(z_t)}{q'(z_t)} \frac{\mu^2}{\sigma^2} \left(1 + \frac{z_t q''(z_t)}{2q'(z_t)} \right) dt - \int_0^T \frac{z_t q''(z_t)}{q'(z_t)} \frac{\mu}{\sigma} dW_t \quad (\text{A.64})$$

Note that the stochastic integral is square-integrable, hence a martingale with mean zero. Moreover, recall that

$$R_t = \begin{cases} -1/q'(z_t) & \text{if } z_t \in [1-\alpha, 1), \\ -z_t^{-1/\gamma}/q'(z_t) & \text{if } z_t \in [1, \infty). \end{cases} \quad (\text{A.65})$$

Then by Ergodic theorem, and the invariant distribution density $\nu(z)$ of z in (A.56), we have:

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}[\log X_T] &= \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T r - R_t - \frac{z_t q''(z_t)}{q'(z_t)} \frac{\mu^2}{\sigma^2} \left(1 + \frac{z_t q''(z_t)}{2q'(z_t)} \right) dt \right] \\ &= \int_{1-\alpha}^1 \left\{ r + \frac{1}{q'(z)} - \frac{zq''(z)}{q'(z)} \frac{\mu^2}{\sigma^2} \left(1 + \frac{zq''(z)}{2q'(z)} \right) \right\} \nu(z) dz \\ &\quad + \int_1^\infty \left\{ r + \frac{z^{-1/\gamma}}{q'(z)} - \frac{zq''(z)}{q'(z)} \frac{\mu^2}{\sigma^2} \left(1 + \frac{zq''(z)}{2q'(z)} \right) \right\} \nu(z) dz \\ &= r - r(1-\alpha)^{\rho+1} \left(\frac{\gamma\rho - \rho - 1}{\gamma\rho} \right) \\ &\quad + (1+\rho)(1-\alpha)^{\rho+1} \int_{1-\alpha}^1 \frac{1}{z^{\rho+1}} \left\{ \frac{1}{q'(z)} - \frac{zq''(z)}{q'(z)} \frac{\mu^2}{\sigma^2} \left(1 + \frac{zq''(z)}{2q'(z)} \right) \right\} dz \end{aligned}$$

□

Proof of Theorem 5.2. The expected wealth return rate is the risk-free interest rate r plus the following term:

$$\int_{1-\alpha}^{\infty} \pi(z) \nu(z) \mu dz \quad (\text{A.66})$$

where ν is the invariant distribution density of z as in (A.56). Since $\pi(z) = -\frac{z q''(z)}{q'(z)} \frac{\mu}{\sigma^2}$, then

$$\int_{1-\alpha}^{\infty} \pi(z) \nu(z) \mu dz = -\frac{2r(1-\alpha)^{\rho+1}(\rho+1)}{\rho} \int_{1-\alpha}^{\infty} \frac{q''(z)}{z^{\rho+1} q'(z)} dz \quad (\text{A.67})$$

The result in the theorem follows. $\square$

Proof of Theorem 5.3. By the homogeneity of $\tilde{V}$ in (8.15), we have

$$\tilde{V}(y, h_0) = h_0^{1-\gamma^*} q(1-\alpha) = \left(\frac{y}{1-\alpha} \right)^{1-1/\gamma^*} q(1-\alpha) \quad (\text{A.68})$$

Therefore, the value y making the dual-equality, $V(x, h_0) = \inf_{y>0} (\tilde{V}(y, h_0) + xy)$, holds is:

$$\hat{y} = \left(-\frac{(1-\alpha)^{1/\gamma^*-1} q(1-\alpha)}{x} (1-1/\gamma^*) \right)^{\gamma^*} \quad (\text{A.69})$$

Since for a general α , the optimal target process is: $h_t = \left(\frac{\hat{y}}{1-\alpha} \right)^{-1/\gamma^*} (\inf_{s \leq t} M_s)^{-1/\gamma^*}$; and by (A.69), it follows that

$$h_t = \frac{x \gamma^*}{(1-\gamma) q(1-\alpha)} \left(\inf_{s \leq t} M_s \right)^{-1/\gamma^*} \quad (\text{A.70})$$

Then

$$\frac{h_t^1}{h_t^2} = \frac{\gamma_1^* q(1-\alpha_2)}{\gamma_2^* q(1-\alpha_1)} \left(\inf_{s \leq t} M_s \right)^{1/\gamma_2^*-1/\gamma_1^*} \leq 1 \quad (\text{A.71})$$

$$\Leftrightarrow \inf_{s \leq t} M_s \leq \left[\frac{\gamma_2^* q(1-\alpha_1)}{\gamma_1^* q(1-\alpha_2)} \right]^{\frac{\gamma_1^* \gamma_2^*}{(\alpha_1 - \alpha_2)(1-\gamma)}} \quad (\text{A.72})$$

$$\Leftrightarrow \inf_{s \leq t} \left[-\left(r + \frac{\mu^2}{2\sigma^2} \right) t - \frac{\mu}{\sigma} W_t \right] \leq \frac{\gamma_1^* \gamma_2^*}{(\alpha_1 - \alpha_2)(1-\gamma)} \log \left[\frac{\gamma_2^* q(1-\alpha_1)}{\gamma_1^* q(1-\alpha_2)} \right] \quad (\text{A.73})$$

$$\Leftrightarrow \sup_{s \leq t} \left[\left(r + \frac{\mu^2}{2\sigma^2} \right) t + \frac{\mu}{\sigma} W_t \right] \geq \frac{\gamma_1^* \gamma_2^*}{(\alpha_1 - \alpha_2)(1-\gamma)} \log \left[\frac{\gamma_1^* q(1-\alpha_2)}{\gamma_2^* q(1-\alpha_1)} \right] \geq 0 \quad (\text{A.74})$$

Equivalently,

$$\begin{aligned} \tau_{\alpha_1, \alpha_2} &= \inf \left\{ t : \left(r + \frac{\mu^2}{2\sigma^2} \right) t + \frac{\mu}{\sigma} W_t \geq \frac{\gamma_1^* \gamma_2^*}{(\alpha_1 - \alpha_2)(1-\gamma)} \log \left[\frac{\gamma_1^* q(1-\alpha_2)}{\gamma_2^* q(1-\alpha_1)} \right] \right\} \\ &= \inf \left\{ t : \left(\frac{r\sigma}{\mu} + \frac{\mu}{2\sigma} \right) t + W_t \geq c \right\} \end{aligned} \quad (\text{A.75})$$

where $c = \frac{\gamma_1^* \gamma_2^* \sigma}{(\alpha_1 - \alpha_2)(1-\gamma) \mu} \log \left[\frac{\gamma_1^* q(1-\alpha_2)}{\gamma_2^* q(1-\alpha_1)} \right] = \frac{\gamma_1^* \gamma_2^* \sigma}{(\alpha_1 - \alpha_2)(1-\gamma) \mu} \log \left(\frac{b_2}{b_1} \right) \geq 0$. The second equality holds since the bliss ratio $b_i = -q'(1-\alpha_i) = -\frac{\gamma_i^*}{(\gamma-1)q(1-\alpha_i)}$, ($i = 1, 2$).

The density and expectation of τ_{target} follow from Borodin and Salminen (2002) (Part II, 2.2, Page 295). $\square$

Theorem A.11. *The weight on the risky asset π_t in (4.6) is increasing in the target-wealth ratio x/h .*

Proof. Recall from (8.19) that the risky asset weight is given by

$$\pi(z) = -\frac{zq''(z)}{q'(z)} \frac{\mu}{\sigma^2} \quad \text{where} \quad q'(z) = -\frac{x}{h} \quad (\text{A.76})$$

and hence

$$\frac{d\pi}{dx} = \frac{d\pi}{dz} \frac{dz}{dx} = \left(\frac{zq''(z)}{q'(z)} \right)' \frac{\mu}{\sigma^2} \frac{1}{hq''(z)} \quad (\text{A.77})$$

Since q is convex, it remains to check that $\left(\frac{zq''(z)}{q'(z)} \right)' > 0$. Since

$$\left(\frac{zq''(z)}{q'(z)} \right)' = -\frac{(\gamma-1)\rho(\rho+1)(\gamma\rho+1)(-\gamma\rho+z^\rho(\rho((\gamma-1)z+1)+1)-1)}{((\gamma-1)z^{\rho+1}-(\gamma\rho+1)(\rho+(\gamma-1)(\rho+1)z))^2} \quad (\text{A.78})$$

it is in turn enough to check that

$$-\gamma\rho+z^\rho(\rho((\gamma-1)z+1)+1)-1 \leq 0 \quad (\text{A.79})$$

for $z \in [1-\alpha, 1]$. (Note that when $z > 1$, the wealth to target ratio is lower than the gloom point, and the weight of the risky asset is the Merton weight.)

This condition is clearly satisfied for $\gamma = 1$, as this expression reduces to $(\rho+1)(z^\rho-1)$. In addition, the expression is decreasing in γ , because its partial derivative is

$$\rho(z^{\rho+1}-1) \leq 0 \quad (\text{A.80})$$

and therefore it is less than or equal to zero for $\gamma \geq 1$. $\square$

Proof of Proposition 6.1. The Merton strategy

$$c_t = mX_t; \quad \pi_t = \frac{\mu}{\gamma\sigma^2} \quad (\text{A.81})$$

leads to the corresponding wealth process

$$X_t = x \exp \left\{ \frac{1}{\gamma} \left(r + \frac{\mu^2}{2\sigma^2} \right) t + \frac{\mu}{\gamma\sigma} W_t \right\}. \quad (\text{A.82})$$

Let $W_t^{(\zeta)} = \zeta t + W_t$, where $\zeta = \frac{r\sigma}{\mu} + \frac{\mu}{2\sigma}$. Denote the corresponding maximum process by $(W_t^{(\zeta)})^* = \sup_{0 \leq s \leq t} W_s^{(\zeta)}$. Then the resulting target spending equals to

$$h_t = \bar{h} \vee mx \exp \left\{ \frac{\mu}{\gamma\sigma} (W_t^{(\zeta)})^* \right\} \quad (\text{A.83})$$

and its utility becomes

$$U_t = \frac{1}{1-\gamma} \left(\frac{c_t}{h_t^\alpha} \right)^{1-\gamma} = \frac{1}{1-\gamma} \left(\frac{mx \exp \left\{ \frac{\mu}{\gamma\sigma} W_t^{(\zeta)} \right\}}{\bar{h}^\alpha \vee (mx)^\alpha \exp \left\{ \frac{\alpha\mu}{\gamma\sigma} (W_t^{(\zeta)})^* \right\}} \right)^{1-\gamma} \quad (\text{A.84})$$

$$= \frac{1}{1-\gamma} \left[\left(\frac{mx}{\bar{h}^\alpha} \right)^{1-\gamma} U_t^{(1)} + (mx)^{1-\gamma} U_t^{(2)} \right], \quad (\text{A.85})$$

where

$$U_t^{(1)} = \exp \left\{ \frac{(1-\gamma)\mu}{\gamma\sigma} W_t^{(\zeta)} 1_{\{(W_t^{(\zeta)})^* < \kappa_0\}} \right\} \quad (\text{A.86})$$

$$U_t^{(2)} = \exp \left\{ \frac{(1-\gamma)\mu}{\gamma\sigma} \left(W_t^{(\zeta)} - \alpha (W_t^{(\zeta)})^* \right) 1_{\{(W_t^{(\zeta)})^* \geq \kappa_0\}} \right\} \quad \kappa_0 = \frac{\gamma\sigma}{\mu} \ln \left(\frac{\bar{h}}{mx} \right) \quad (\text{A.87})$$

Then the expected utility of the Merton strategy is

$$U^M(x, \alpha, \gamma, h) = \mathbb{E} \int_0^\infty \frac{1}{1-\gamma} \left[\left(\frac{mx}{\bar{h}^\alpha} \right)^{1-\gamma} U_t^{(1)} + (mx)^{1-\gamma^*} U_t^{(2)} \right] dt \quad (\text{A.88})$$

$$= \frac{1}{1-\gamma} \int_0^\infty \left(\frac{mx}{\bar{h}^\alpha} \right)^{1-\gamma} \mathbb{E} U_t^{(1)} + (mx)^{1-\gamma^*} \mathbb{E} U_t^{(2)} dt \quad (\text{A.89})$$

Applying Corollary A.7, it follows that

$$\mathbb{E} U_t^{(1)} = \mathbb{E} \exp \left\{ \frac{(1-\gamma)\mu}{\gamma\sigma} W_t^{(\zeta)} \right\} - \mathbb{E} \exp \left\{ \frac{(1-\gamma)\mu}{\gamma\sigma} W_t^{(\zeta)} 1_{\{(W_t^{(\zeta)})^* > \kappa_0\}} \right\} \quad (\text{A.90})$$

$$= e^{-mt} \left[\Phi \left(-\kappa_1 \sqrt{t} + \kappa_0 / \sqrt{t} \right) - e^{2\kappa_0 \kappa_1} \Phi \left(-\kappa_1 \sqrt{t} - \kappa_0 / \sqrt{t} \right) \right] \quad (\text{A.91})$$

$$\mathbb{E} U_t^{(2)} = \frac{2\kappa_2}{\kappa_1 + \kappa_2} \exp \left\{ -mt - \frac{(1-\gamma)\alpha\mu}{2\gamma\sigma} (\kappa_1 + \kappa_2)t \right\} \Phi \left(\kappa_2 \sqrt{t} - \kappa_0 / \sqrt{t} \right) \quad (\text{A.92})$$

$$+ \frac{2\kappa_1}{\kappa_1 + \kappa_2} \exp \left\{ \kappa_0 (\kappa_1 + \kappa_2) - mt \right\} \Phi \left(-\kappa_1 \sqrt{t} - \kappa_0 / \sqrt{t} \right) \quad (\text{A.93})$$

where

$$\kappa_1 = \frac{r\sigma}{\mu} + \frac{(2-\gamma)\mu}{2\gamma\sigma}, \quad \kappa_2 = \frac{r\sigma}{\mu} + \frac{(2-2\gamma^* + \gamma)\mu}{2\gamma\sigma} \quad (\text{A.94})$$

To calculate the expected utility in (A.89) it remains to evaluate the following integrals by parts:

$$\int_0^\infty \mathbb{E} U_t^{(1)} dt = \frac{1}{m} - \frac{\kappa_0 \exp \left\{ \kappa_0 \left(\kappa_1 - \frac{r\sigma}{\mu} - \frac{\mu}{2\sigma} \right) \right\}}{\sqrt{2\pi}m} \int_0^\infty \left(\frac{1}{\sqrt{t^3}} \right) \exp \left\{ -\frac{1}{2} \left[\left(\frac{r\sigma}{\mu} + \frac{\mu}{2\sigma} \right) \sqrt{t} - \frac{\kappa_0}{\sqrt{t}} \right]^2 \right\} dt \quad (\text{A.95})$$

$$\int_0^\infty \mathbb{E} U_t^{(2)} dt = \frac{2 \exp \left\{ \kappa_0 \left(\kappa_2 - \frac{r\sigma}{\mu} - \frac{\mu}{2\sigma} \right) \right\}}{\sqrt{2\pi}(\kappa_1 + \kappa_2)} \int_0^\infty \left(\left(\frac{\kappa_2^2}{\tilde{m}} - \frac{\kappa_1^2}{m} \right) \frac{1}{2\sqrt{t}} + \left(\frac{\kappa_2}{\tilde{m}} + \frac{\kappa_1}{m} \right) \frac{\kappa_0}{2\sqrt{t^3}} \right) \exp \left\{ -\frac{1}{2} \left[\left(\frac{r\sigma}{\mu} + \frac{\mu}{2\sigma} \right) \sqrt{t} - \frac{\kappa_0}{\sqrt{t}} \right]^2 \right\} dt \quad (\text{A.96})$$

with $\tilde{m} = \frac{\gamma^* - 1}{\gamma} \left(r + (1 - \alpha + \alpha\gamma) \frac{\mu^2}{2\gamma\sigma^2} \right)$. To compute the two integrals on the right hand side of

the above equations, let $u = \left(\frac{r\sigma}{\mu} + \frac{\mu}{2\sigma} \right) \sqrt{t} - \frac{\kappa_0}{\sqrt{t}}$, then $\sqrt{t} = \frac{u + \sqrt{u^2 + 4\kappa_0 \left(\frac{r\sigma}{\mu} + \frac{\mu}{2\sigma} \right)}}{2 \left(\frac{r\sigma}{\mu} + \frac{\mu}{2\sigma} \right)}$, and

$$\frac{dt}{\sqrt{t^3}} = \left(\frac{1}{\kappa_0} - \frac{u}{\kappa_0 \sqrt{u^2 + 4\kappa_0 \left(\frac{r\sigma}{\mu} + \frac{\mu}{2\sigma} \right)}} \right) du \quad \frac{dt}{\sqrt{t}} = \frac{1}{\frac{r\sigma}{\mu} + \frac{\mu}{2\sigma}} \left(1 + \frac{u}{\sqrt{u^2 + 4\kappa_0 \left(\frac{r\sigma}{\mu} + \frac{\mu}{2\sigma} \right)}} \right) du \quad (\text{A.97})$$

It follows that

$$\int_0^\infty \left(\frac{1}{\sqrt{t^3}} \right) \exp \left\{ -\frac{1}{2} \left[\left(\frac{r\sigma}{\mu} + \frac{\mu}{2\sigma} \right) \sqrt{t} - \frac{\kappa_0}{\sqrt{t}} \right]^2 \right\} dt = \int_{-\infty}^\infty \exp \left\{ -\frac{u^2}{2} \right\} \left(\frac{1}{\kappa_0} - \frac{u}{\kappa_0 \sqrt{u^2 + 4\kappa_0 \left(\frac{r\sigma}{\mu} + \frac{\mu}{2\sigma} \right)}} \right) du = \frac{\sqrt{2\pi}}{\kappa_0}$$

$$\int_0^\infty \left(\frac{1}{\sqrt{t}} \right) \exp \left\{ -\frac{1}{2} \left[\left(\frac{r\sigma}{\mu} + \frac{\mu}{2\sigma} \right) \sqrt{t} - \frac{\kappa_0}{\sqrt{t}} \right]^2 \right\} dt = \frac{1}{\frac{r\sigma}{\mu} + \frac{\mu}{2\sigma}} \int_{-\infty}^\infty \exp \left\{ -\frac{u^2}{2} \right\} \left(1 + \frac{u}{\sqrt{u^2 + 4\kappa_0 \left(\frac{r\sigma}{\mu} + \frac{\mu}{2\sigma} \right)}} \right) du = \frac{\sqrt{2\pi}}{\frac{r\sigma}{\mu} + \frac{\mu}{2\sigma}}$$

Hence, by (A.95) and (A.96)

$$\int_0^\infty \mathbb{E} U_t^{(1)} dt = \frac{1 - \exp \left\{ \kappa_0 \left(\kappa_1 - \frac{r\sigma}{\mu} - \frac{\mu}{2\sigma} \right) \right\}}{m} \quad (\text{A.98})$$

$$\int_0^\infty \mathbb{E} U_t^{(2)} dt = \frac{\exp \left\{ \kappa_0 \left(\kappa_2 - \frac{r\sigma}{\mu} - \frac{\mu}{2\sigma} \right) \right\}}{(1 - \alpha)m} \quad (\text{A.99})$$

Plugging these expressions into (A.89) leads to an expected utility equal to:

$$U^M(x, \alpha, \gamma, h) = \frac{(x\bar{h}^{-\alpha})^{1-\gamma}}{1-\gamma} m^{-\gamma} \left(1 + \frac{\alpha}{1-\alpha} \left(\frac{\bar{h}}{mx} \right)^{1-\gamma} \right). \quad (\text{A.100})$$

(For $\bar{h} = mx$, this expression reduces to $\frac{x^{1-\gamma^*}}{1-\gamma^*} m^{-\gamma^*}$.) To obtain the equivalent relative loss, recall that the value function has the expression

$$U^{OP}(x, \alpha, \gamma, h) = V(x, \bar{h}) = \bar{h}^{(1-\gamma)(1-\alpha)} \inf_{z>0} (q(z) + \frac{x}{\bar{h}} z) \quad (\text{A.101})$$

and hence

$$V(x, h) = h^{(1-\gamma)(1-\alpha)} (q(z) - q'(z)z) \quad \text{where } z \text{ satisfies } q'(z) = -\frac{x}{h} \quad (\text{A.102})$$

Setting the above expression for $V(x, h)$ equal to (A.100), equation (6.2) follows, while the expression in (6.1) results from replacing x with $x(1 - L)$ in the first-order condition $q'(z) = -\frac{x}{h}$ and solving for L . $\square$

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