Valuation of Highly Leveraged Firms

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The financial policy highly leveraged firms (HLFs) commonly follow implies uncertain leverage. Explicit allowance for this characteristic leads to two complementary pricing models. A recursive formula for the value of HLF follows from applying the adjusted present value (APV) approach to uncertain tax shields. This formula is used to evaluate the robustness of the simple APV rule and other valuation approaches used in practice. The HLF equity is also modeled as a call option with uncertain exercise price, which provides a natural way of dealing with uncertain leverage and complements the APV approach. Both models require inputs that are usually observable. The models developed in this article apply to the valuation of firms undergoing financial restructuring, as well as to leveraged buyouts and project financing. In all these situations, firms deploy all or a significant portion of their free cash flow to debt reduction and their leverage ratio is uncertain.

Many highly leveraged firms (HLFs) plan to reduce their leverage ratios over time toward targets by deploying their free cash flow to debt reduction. These firms include leveraged buyouts (LBOs) and recapitalizations, such as Owens Corning Fiberglass and Southland, as well as firms such as Eastman Kodak and K-Mart, which for a variety of reasons, ended up with a leverage higher than their targets and embarked on financial restructuring. Valuation of such firms is difficult because their future leverage ratios are uncertain. Another valuation problem with uncertain debt reduction to which the models discussed in this article apply is project financing.

The usual valuation approaches are unable to deal with the special characteristic of HLFs: Debt reduction is a function of cash flow realizations and, therefore, is uncertain.[1] Leverage changes invalidate the use of a constant weighted-average cost of capital or a constant cost of equity. Furthermore, available formulas for adjusting the cost of equity as a function of leverage are not applicable to this case. For example, Miles and Ezzell (1985) pointed out that relevering beta coefficients using the well-known formula of Hamada (1972) is only valid when future cash flows are a perpetuity.

Predetermined leverage changes offer no diffi-

culty because the appropriate valuation of the firm can be made by applying Myers' (1974) adjusted present value (APV) rule. This possibility was noted by Inselbag and Kaufold (1989), who proposed using APV to value LBOs with predetermined debt levels. Kaplan and Rubback (1995) also used APV in their empirical study of valuation methods. Myers' original APV rule, however, does not strictly apply when debt reduction is uncertain. Specifically, the correct APV rule for HLFs needs to allow for uncertain leverage in valuing the tax shield, which is not done when discounting the tax shield at the cost of debt.[2]

In this article, we examine two approaches to the valuation of HLFs that explicitly account for uncertain leverage. First, we develop a recursive APV valuation formula for the value of the firm and compare it with alternative valuation methods. In particular, we examine the robustness of the simple APV rule, which assumes predetermined debt levels. Although the simple APV performs almost as well as recursive APV, other commonly used valuation methods can lead to significant error. Another approach to HLF valuation is to treat equity as a call option on the value of the firm. We develop an option pricing model and illustrate its use. The option pricing approach is shown to be conceptually more appealing than APV because it avoids explicit valuation of the tax shield. In practice, however, using both APV and option pricing is likely to provide better valuations of HLFs.

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A RECURSIVE APV MODEL

The financial policy modeled in this article is that of a firm planning to reduce its leverage ratio over a number of periods by applying its free cash flow to debt reduction. This reduction is done in practice by retiring outstanding debt or reducing the refinancing of maturing debt. This gradual process of debt reduction, rather than a single-step recapitalization, is consistent with Myers' pecking order theory and with the cash flow signaling hypothesis of leveraged buyouts in which LBO firms revert to more conventional financial policies in the years following the buyout (Myers 1984, Arzac 1992).

Debt reduction in any given period is then a function of the random cash flow realization of that period. In this section, cash flows are partitioned and valued according to value additivity as in Myers (1974). The result is a recursive equation for the value of the firm. The Modigliani and Miller (1958) risk-class assumption provides a justification for the valuation approach. A more explicit rationale is provided by the multiperiod capital asset pricing model under the assumptions specified by Fama (1977) and Constantinidis (1980).

The following notation is used:

Z_t = the free cash flow of the unlevered firm at time t (i.e., the cash flow after taking into account depreciation, capital expenditures, and taxes)

ρ = the discount rate for the unlevered cash flows

B_t = the outstanding debt at time t

r = the coupon and discount rates for debt

r_f = the risk-free rate of interest

τ = the corporate income tax rate

Y_t = the cash flow of the levered firm

$P_i(Y_t)$ = the value of Y_t at time i

$P_{i,k}$ = $\Sigma_{t=i+1,k} P_i(Y_t)$, the value of $Y_{i+1},...,Y_k$ at time i

V_i = $\Sigma_{t=i+1,\infty} P_i(Y_t)$, the value of the firm at time i

S_i = $V_i - B_i$, the value of equity at time i

L = B_t/V_t, the leverage ratio for $t \geq T$

r_L = the interest rate on debt when leverage is L

δ = dividend as a fraction of Y_t

Two commonly made assumptions are adopted in this section: ρ, r, and r_L are constant over time, and the risk characteristics of debt and the tax shield conditional on the level of debt are the same. Although the model below is still valid for known time-changing rates and different rates

of discount for debt and the conditional tax shield, these assumptions recognize the information constraints often faced in valuation. The option pricing model presented below does not require r as an input or explicit valuation of the tax shield.

The financial policy of the HLF can be stated as follows: All the cash flow, Y_t, from $t = 1$ to T is dedicated to paying interest and to debt reduction, and the firm is recapitalized with a constant leverage ratio L at time T. It is shown in the appendix that the value of the firm under this policy is

$$V_0 = P_{0,T} + P_0(V_T), \tag{1}$$

where the value of the first T cash flows $Y_1,...,Y_T$ is computed recursively from

$$P_{0,t} = P_{0,t-1} + \frac{E_0(Z_t)}{(1+\rho)^t} + \frac{\tau r}{1+r}(B_0 - P_{0,t-1}),$$
$$t = 2,...,T. \tag{2}$$

In order to obtain the continuing value V_T in Equation 1, Miles and Ezzell's (1980) result applies because the firm will maintain a constant leverage ratio L from time T onward. Thus, V_T is obtained by discounting the post-time-T cash flows at the constant rate $w = \rho - \tau r_L L (1 + \rho)/(1 + r)$.[3]

Equations 1 and 2 provide a simple way to compute the value of an HLF. First, Equation 2 is solved recursively forward in time from Time 1 to T in order to obtain the value of the first T cash flows, $P_{0,T}$. The present value of continuing value at time T is then added to $P_{0,T}$. When the post-time-T cash flows can be assumed to grow at a constant rate g, Equation 1 becomes:[4]

$$V_0 = P_{0,T} + \frac{E_0(Z_{T+1})}{(1+\rho)^T(w-g)}. \tag{3}$$

A closed-form solution to difference Equation 2 can be obtained for special cases such as when $E_0(Z_t)$ grows at a constant rate g. Then, $E_0(Z_t) = E_0(Z_1)(1 + g)^{t-1}$, and Equation 3 becomes

$$V_0 = \left[\frac{p^T - a^T}{1 + \rho\rho - a} + \frac{p^t}{w - g}\right]E_0(Z_1) + (1 - a^T)B_0, \tag{4}$$

where $p = (1 + g)/(1 + \rho)$ and $a = (1 - \delta)[1 + (1 - \tau)r]/(1 + r)$.

Example 1: Consider the valuation of an HLF that starts with an initial leverage ratio of 88 percent and plans to reduce it to 35 percent via a recapitalization at the end of Year 5. Until then, all free cash flow will go to debt reduction (i.e., $\delta = 0$). Table 1 contains the data assumed for this example, and Table 2

presents the computation of Equations 2 and 3. The closed-form formula can also be used in the present case. Substituting the parameter values into Equation 4 yields $V_0 = \$1,706.60$.

Table 1. Data for Recursive APV Valuation

$E_0(Z_1)$	$150
g	4.0%
B_0	$1,500
L	35.0%
τ	40.0%
r	11.0%
r_L	9.0%
r	15.0%

RECURSIVE APV VERSUS ALTERNATIVE APPROACHES

This section addresses the error committed by applying simple APV to HLF valuation. The simple APV rule is defined as follows:

$$APV = \Sigma_{t=1,T}\frac{E_0(Z_t)}{(1+\rho)^t} + \Sigma_{t=1,T}\frac{\tau r B_{t-1}}{(1+r)^t} +$$

$$\frac{E_0(Z_{T+1})}{(1+\rho)^T(w - g_{T+1})}, \qquad (5)$$

where $B_t = B_{t-1} - (E_0(Y_t) - rB_{t-1})$, $E_0(Y_t) = E_0(Z_t) + \tau r B_{t-1}$, for $t = 1, \dots, T$.

Equation 5 yields a lower tax shield value than Equation 3 because it discounts all the components of the tax shield at the cost of debt, ignoring that uncertain debt reduction has lower present value.

Table 3 presents the percent error of underestimation committed by Equation 5 for different values of $\Delta = \rho - r$ and T. The remaining parameters are as stated in Table 1. Panel A exhibits the error committed in valuing the tax shield. This error exceeds 5 percent when the recapitalization period is 10 years and $\Delta \geq 3$ percent. Panel B exhibits the error committed in valuing equity

Table 3 shows that simple APV results in a significant error only when the spread, Δ, is large and the changes in leverage exceed five years. This means that simple APV does not lead to significant error in the highest leverage cases such as LBOs for which r is close to ρ. The error is also small for short-period recapitalizations, even when $\rho \gg r$. This case includes non-LBO firms that reduce their leverage from, say, 40 percent to 30 percent, as Dupont did during the 1980s. The conclusion is that simple APV is rather robust to leverage uncertainty.

One may think that a further improvement over simple APV can be attained by discounting the risky tax shield at ρ, that is, by substituting ρ for r in the denominator of the second term of Equation 5.[5] The errors committed by this approach are summarized in Table 4, which shows that discounting all the components of the tax shield at the higher rate, ρ, significantly increases the valuation error and is not a suitable alternative to recursive or simple APV. One problem with this approach is that it ignores that, under uncertain debt reduction, the tax shield is negatively correlated to the unlevered cash flows. In fact, it is straightforward to show that $cov[\tau r B_t, Z_t] = -\tau r\, var[Z_t]$. This means that the tax shield has to be discounted at a rate lower than ρ.

ON DISCOUNTING THE EQUITY CASH FLOWS

An alternative approach sometimes used in valuation is discounting the cash flow corresponding to equity holders (i.e., the cash flow after allowing for interest on debt and the effect of financial transactions) at the cost of equity of the firm. In valuing HLF, it is common practice to obtain the internal rate of return of the cash flow to equity

Table 2. Recursive APV Computation

t	0	1	2	3	4	5
$E_0(Z_t)$		$150.0	$156.0	$162.2	$168.7	$175.5
$V_{0,t}$		189.9	359.8	511.7	647.3	768.4
V_T						1,887.1
$V_{0,T}$	$768.4					
$V_0(V_T)$	938.2					
$V_{0,T} + V_0(V_T)$	1,706.6					
S_0	206.6					

Table 3. Percent Valuation Errors from Misspecification of the APV Formula (I)

| | Recapitalization Period (years) | | | | | |
| | A. Tax Shield | | | B. Equity | | |
Δ	5	7	10	5	7	10
1.0	0.4%	0.8%	1.8%	0.1%	0.2%	0.4%
2.0	0.7	1.6	3.5	0.2	0.5	1.2
3.0	1.1	2.3	5.1	0.4	1.0	2.5
4.0	1.4	3.0	6.5	0.8	2.0	4.9
5.0	1.8	3.6	7.8	1.8	4.1	9.6

Notes: $\Delta = \rho - r$ (%). This table shows the percent underestimation error produced by Equation 5 with respect to Equation 3.

for a given purchase price and compare it with a required return on equity or to use the latter to obtain the present value of the cash flows. The problem with this approach is that the proper definition of the HLF cost of equity needs to allow for changes in the systematic risk of the cash flows over time, which is a function of uncertain leverage. Sometimes, attempts are made to adjust the cost of equity for changes in leverage via Hamada's formula for the beta coefficient of the levered firm (see Hamada 1972). That method leads to significant error.

Table 5 shows the valuation error committed when the cost of equity of the HLF is adjusted over time by relevering β according to Hamada's formula:

$$\beta_{L,t} = \left[1 + (1-\tau)\frac{B_{t-1}}{S_{t-1}}\right]\beta_U,$$

where $\beta_{L,t}$ is the levered β coefficient for period t and β_U is the unlevered β coefficient.

The value of equity is computed by backward iteration starting from the terminal value computed as in Equation 3. The levered β, the debt-to-equity ratio, and the value of equity in each period are determined simultaneously.

Correct discounting of the equity cash flow requires a definition of the cost of equity consistent with valuation theory. Such a definition is available under the assumption of constant debt used by Modigliani and Miller (1958) and the assumption of constant leverage ratio studied by Miles and Ezzell (1980).[6] On the other hand, no practical definition of the future systematic risk and the cost of equity is possible in the case of uncertain debt reduction, because future leverage depends on preceding realizations of the firm's cash flows. Once V_0 has been obtained via Equation 3, however, the *average* cost of equity for the first T periods can be defined as

$$\rho^* = \left(\frac{V_T - B_T}{V_0 - B_0}\right)^{1/T} - 1. \tag{6}$$

In the case of Example 1, $\rho^* = 25.8$ percent. By construction, discounting the equity cash flows at ρ^* gives the correct value of equity.

VALUING HLF EQUITY AS AN OPTION

This section examines an alternative partition of

Table 4. Percent Valuation Errors from Misspecification of the APV Formula (II)

| | Recapitalization Period (years) | | | | | |
| | A. Tax Shield | | | B. Equity | | |
Δ	5	7	10	5	7	10
1.0	2.8%	3.8%	5.3%	0.5%	0.8%	1.3%
2.0	5.4	7.3	10.2	1.3	2.2	3.5
3.0	8.0	10.6	14.6	2.9	4.6	7.3
4.0	10.4	13.8	18.7	5.9	9.1	14.1
5.0	12.7	16.7	22.5	13.3	19.1	27.8

Notes: $\Delta = \rho - r$ (%). This table shows the percent underestimation error produced by Equation 5 with respect to Equation 3 when the tax shield terms of Equation 5 are discounted at ρ.

Table 5. Percent Valuation Errors from Misspecification of Cash-Flow-to-Equity Discounting

| | Recapitalization Period (years) | | | | | |
| | A. Tax Shield | | | B. Equity | | |
Δ	5	7	10	5	7	10
1.0	40.3%	34.6%	29.7%	6.8%	7.1%	7.2%
2.0	39.9	35.7	33.1	9.7	10.5	11.4
3.0	37.2	34.2	33.4	13.4	14.7	16.7
4.0	33.2	31.5	32.0	18.9	20.7	24.1
5.0	28.4	27.6	29.8	29.8	31.5	36.8

Notes: $\Delta = \rho - r$ (%). This table shows the percent underestimation error produced by discounting the cash flow to equity. The beta coefficient of the terminal value is that implicit in w computed from Table 1, for a risk-free rate equal to 8 percent and a market risk premium equal to 6.0 percent.

the HLF cash flow that does not require direct valuation of the tax shield. This partitioning is accomplished by valuing the HLF equity as an option. The option pricing model can be particularly useful when the discount rate for debt is not readily available, as when debt is privately placed with a below-market coupon plus an equity kicker.[7] It provides an alternative to the common practice of using the yield to maturity of debt for valuing the tax shield, which may not be correct when the cost of debt changes over time.[8]

Under the financial policy stated above, for the case in which no dividends are paid, the cash flow of the HLF can be partitioned into the cash flow bondholders receive,

$$\{Y_1, Y_2, ..., Y_{T-1}, Y_T + \min[B_T, V_T]\}, \qquad (7)$$

and the cash flow shareholders receive,

$$\{0, 0, ..., 0, \max[V_T - B_T, 0]\}, \qquad (8)$$

where, as before, $B_T = B_0 - \Sigma_{t=1,T}(Y_t - rB_{t-1})$ and V_T is the continuing value of the firm at time T, which, assuming constant cash flow growth after time T, is $V_T = E_0(Z_{T+1})/(w - g)$.

It follows from Equations 7 and 8 that

$$B_0 = V_0 - c(V_T, B_T, T) \qquad (9)$$

and

$$S_0 = c(V_T, B_T, T), \qquad (10)$$

where c is a call option on the value of the firm V_T with exercise price B_T and expiration date T.

Equation 9 is the well-known expression for the value of risky debt noted by Black and Scholes (1973) and further developed by Merton (1973, 1977).[9] This characterization is particularly appropriate for HLFs in which bondholders

receive all the available free cash flow until a recapitalization takes place.[10] For the typical HLF, a significant portion of the outstanding debt becomes due at some time T, which is the natural time for the equityholders' call option to expire. The tax shield is valued in Equation 10 through its contribution to reducing the exercise price B_T.

The model makes the simplifying assumption that equityholders cannot default on their debt prior to T, and therefore, it assumes that the firm can refinance interim cash shortfalls (in particular, unpaid interests are added to the outstanding debt when $Y_t < r(B_{t-1})$. This assumption is not unrealistic because, in practice, HLFs can tap credit lines or additional subordinated financing in order to cover temporary cash shortfalls.[11]

Equation 10 is a call with uncertain exercise price because B_T depends on the previous realizations of the cash flow. Equation 10 can also be interpreted as an option to exchange B_T for V_T. Margrabe (1978) and Fisher (1978) have shown that the Black and Scholes formula permits valuing this type of call when applied to a suitable transformation of the variables. Their result applies to the present case under the standard assumptions leading to the Black–Scholes formula as set out in Merton (1973). It also holds under the assumptions leading to Rubinstein's (1976) discrete-time general equilibrium valuation approach, which is consistent with the discrete-time nature of the present model and does not depend on the creation of a riskless hedge. When applied to Equation 10, the Fisher–Margrabe formula becomes:

$$c(V_T, B_T, T) = P_0(V_T)N(d_1) - P_0(B_T)N(d_2), \qquad (11)$$

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where $N(.)$ is the standard cumulative normal distribution

$$d_1 = \frac{\ln [P_0(V_T)/P_0(B_T)] + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad (12)$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

and

$$\sigma^2 = \sigma_1^2 - 2\sigma_{12} + \sigma_2^2 \,.^{12}$$

To implement Equation 11, one needs to specify the stochastic processes driving V_T and B_T. Because uncertainty of free cash flows is the single source of volatility in the present model, the variance of Z_t imparts volatility and covariance to V_T and B_T. To obtain the latter, one must first model the process generating the free cash flows and the process of expectation revision and then derive the first two moments of the joint distribution of (V_T, B_T). This derivation is shown in the appendix, where the lognormal distribution of (V_T, B_T) is fully characterized under the assumption that V and B follow Ito processes. The only additional input needed to compute Equation 11 is the volatility of the rate of return of the unlevered firm.

Example 2: The equity of the HLF considered in Example 1 is now valued using Equation 11 for $r_f = 8.0$ percent and a volatility of the unlevered return $\sigma_U = 30.0$ percent. Table 6 presents the computations of Equation A10 to Equation A12 and Equation A18 to Equation A22 of the appendix.

Equation 11 yields a value of equity of $S_0 = \$195.4$. As one would expect, this value is somewhat sensitive to changes in the volatility estimate. For the present example, a ±0.01 (±1 percent) change in σ_U results in a $\pm0.025S_0$ change in S_0. Figure 1 depicts the sensitivity of the value of equity to changes in volatility in terms of the semi-elasticity $S_0^{-1}\delta S_0/\delta\sigma_U$.

CONCLUSION

APV and option pricing need the same number of inputs: Option pricing requires volatility as an input, but it does not require knowledge of the rate of discount for the tax shield because it avoids its explicit valuation. In this regard, option pricing is more appealing.

Although recursive APV provides a useful benchmark against which to evaluate the various discounted cash flow approaches, in most cases, it does not result in a significant improvement over simple APV. On the other hand, the cash-flow-to-equity approach with beta relevered at each period can lead to significant error and should be avoided.

In conclusion, either recursive or simple APV applied in conjunction with option pricing seems to offer the best approach to the valuation of highly leveraged firms.

Table 6. Option Pricing of HLF Equity

t	0	1	2	3	4	5
$E_0(Z_t)$		$150.0	$156.0	$162.2	$168.7	$175.5
rB_{t-1}		150.0	144.0	137.0	129.0	119.9
τrB_{t-1}		60.0	57.6	54.8	51.6	48.0
$E_0(Z_t)-(1-\tau)rB_{t-1}$		60.0	69.6	80.0	91.3	103.5
$E_0(B_t)$	$41,500.0	1,440.0	1,370.4	1,290.4	1,199.1	1,095.5
$E_0(V_t)$		—	—	—	—	1,887.1
$P_0(Y_t)$		186.0	167.7	151.1	136.1	122.6

$P_0(B_T)$	$848.9	σ_1^2			0.00234
$P_0(V_T)$	$938.2	σ_2^2			0.01317
$(1+\rho)^T$	2.011	σ_{12}			−0.00939
$E_0(B_T)/P_0(B_T)$	1.291	$\sigma = (\sigma_1^2 - 2\sigma_{12} + \sigma_2^2)^{1/2}$			18.517%
$\omega = \sigma_U/(1+\rho)$	26.09%	S_0			$195.4
$(1+\rho)^{2T}\omega^2$	0.275				
$var_0(B_T)/P_0(B_T)^2$	0.019				
$cov_0(V_T,B_T)/[P_0(B_T)P_0(V_T)]$	−0.029				

Figure 1. Sensitivity of Equity Value to Volatility Estimate

Figure 1. Sensitivity of Equity Value to Volatility Estimate
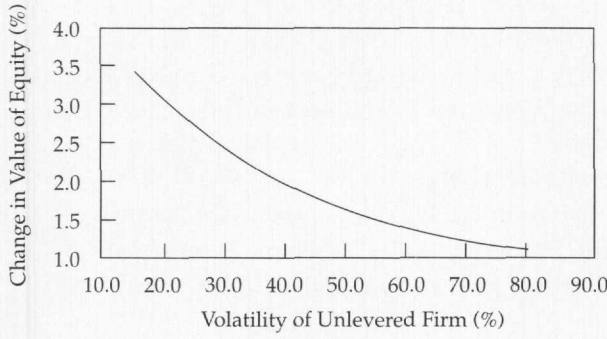
APPENDIX

Recursive APV

The cash flow of the levered firm is

$$Y_t = Z_t + \tau r B_{t-1} \tag{A1}$$

and, since the firm expects to reduce debt by the amount $Y_t - rB_{t-1}$ at time $t = 1,...,T$, successive substitutions of this expression into B_{t-1} yield

$$B_{t-1} = (1+r)^{t-1}B_0 - \Sigma_{j=1,t-1}(1+r)^{t-j-1}Y_j, \tag{A2}$$

and

$$Y_t = Z_t + \tau r[(1+r)^{t-1}B_0 - \Sigma_{j=1,t-1}$$

$$(1+r)^{t-j-1}Y_j], t = 2,...,T, \tag{A3}$$

where $Y_1 = Z_1 + \tau r B_0$.

In order to obtain the present value of Equation A3, one first notes that the uncertainty about the cash flow $(1+r)^{t-j-1}Y_j$ on the right-hand side of Equation A3 is resolved at time j and should be discounted at r from $t-1$ to j. Thus, $P_0[\Sigma_{j=1,t-1}$ $(1+r)^{t-j-1}Y_j] = (1+r)^{-1}\Sigma_{j=1,t-1}P_0(Y_j)$ and

$$P_0(Y_t) = \frac{E_0(Z_t)}{(1+\rho)^t} + \frac{\tau r}{1+r}(B_0 - P_{0,t-1}), t = 2,...,T, \tag{A4}$$

where $E_0(Z_t)$ is the time-zero expectation of Z_t, and $P_{0,1} = P_0(Y_1) = E_0(Z_1)(1+\rho)^{-1} + \tau r B_0(1+r)^{-1}$.

Summing Equation A4 over t gives the present value of the first t cash flows.

$$P_{0,t} = P_{0,t-1} + \frac{E_0(Z_t)}{(1+\rho)^t} + \frac{\tau r}{1+r}(B_0 - P_{0,t-1}),$$

$$t = 2,...,T. \tag{A5}$$

This result can be easily extended to a firm with positive dividend payout. Let dividend at time t be a fraction δ of the levered cash flow Y_t.[13] Then, debt reduction at time t is $(1-\delta)Y_t - rB_{t-1}$ and Equation A4 becomes

$$P_{0,t} = P_{0,t-1} + \frac{E_0(Z_t)}{(1+\rho)^t} + \frac{\tau r}{1+r}[B_0 - (1-\delta)P_{0,t-1}],$$

$$t = 2,...,T. \tag{A6}$$

Adding to $P_{0,T}$ the present value of the value of the firm at time T V_T (the continuing value) yields the value of the firm at time 0:

$$V_0 = P_{0,T} + P_0(V_T). \tag{A7}$$

Modeling the Return Distribution

The following parsimonious characterization of Z_t and $E_j(Z_t)$ is assumed:

$$Z_t = E_{t-1}(Z_t)u_t, \tag{A8}$$

where u_t is a random variable with $E_j(u_t) = 1$, $var_j(u_t) = \omega^2$, $cov_j(u_k,u_t) = 0 \ \forall \ j, k < t$. Expectations are revised as a function of cash flow realizations according to

$$E_j(Z_t) = \frac{E_{j-1}(Z_t)Z_j}{E_{j-1}(Z_j)}, t = j+1,... \tag{A9}$$

Equations A8 and A9 permit deriving the following expressions for the covariances of the distribution of (V_T, B_T):

$$var_0(V_T) = (w-g)^{-2}E_0(Z_{T+1})^2\omega^2, \tag{A10}$$

$$var_0(B_T) = \omega^2\Sigma_{1,T}[(\tau r)^2 + (1+r)^2]^{t-1}E_0(Z_t)^2, \tag{A11}$$

$$cov_0(V_T,B_T) = -(w-g)^{-1}E_0(Z_T)E_0(Z_{T+1})\omega^2. \tag{A12}$$

Furthermore, it is assumed that the distribution of (V_T, B_T) is well approximated by a lognormal distribution characterized by these moments. ω can be estimated from the volatility, σ_U, of the rate of return of the unlevered firm. In fact, Equations A8 and A9 imply $\sigma_U = (1+\rho)\omega$.

The values of V_T and B_T at time zero are

$$P_0(V_T) = \frac{E_0(Z_{T+1})}{(1+\rho)^T(w-g)}, \tag{A13}$$

$$P_0(B_T)\frac{(1+r)^T}{(1+r_f)^T}B_0 - \Sigma_{t=1,T}\frac{(1+r)^{T-t}}{(1+r_f)^{T-t}}P_0(Y_t), \tag{A14}$$

where

$$P_0(Y_t) \frac{E_0(Z_t)}{(1+\rho)^t} + \tau r \left[\frac{(1+r)^{t-1}}{(1+r_f)^t} B_0 - \right.$$

$$\left. \Sigma_{j=1,t-1} \frac{(1+r)^{t-j-1}}{(1+r_f)^{t-j}} P_0(Y_j) \right] \quad \text{for } t > 1, \tag{A15}$$

and

$$P_0(Y_1) = E_0(Z_1)/(1+\rho) + \tau r B_0/(1+r_f).^{14}$$

Equations A14 and A15 follow from Equations A2 and A3, respectively. Furthermore, $P_T(V_T) = V_T$ and $P_T(B_T) = B_T$.

$P_t(V_T)$ and $P_t(B_T)$ are assumed to follow the Ito processes:

$$dP(V_T) = \mu_1 P(V_T)dt + \sigma_1 P(V_T)dz_1, \tag{A16}$$

$$dP(B_T) = \mu_2 P(B_T)dt + \sigma_2 P(B_T)dz_2, \tag{A17}$$

with $\sigma_{12} = cov(dz_1, dz_2)$. μ_1, μ_2, σ_1, σ_2, and σ_{12} are the moments of the normal distribution of (dz_1, dz_2) such that, under Equations A16–A17, the moments of (V_T, B_T) equal $E_0(V_T)$, $E_0(B_T)$, and Equations A10–A12 above.[15] Therefore, σ_1, σ_2,

and σ_{12} can be solved as a function of ω from the following equations for the moments of the log-normal distribution:

$$\exp(\mu_1 T) = (1+\rho)^T, \tag{A18}$$

$$\exp(\mu_2 T) = \frac{E_0(B_T)}{P_0(B_T)}, \tag{A19}$$

$$\exp(2\mu_1 T)[\exp(\sigma_1^2 T) - 1] = (1+\rho)^{2T}\omega^2. \tag{A20}$$

$$\exp(2\mu_2 T)[\exp(\sigma_2^2 T) - 1] = \frac{\omega^2}{[P_0(B_T)]^2}\Sigma_{1,T}[(\tau r^2) +$$

$$(1+r)^2]^{t-1} E_0(Z_t)^2, \tag{A21}$$

$$\exp[(\mu_1 + \mu_2)T][\exp(\sigma_{12}T) - 1)] =$$

$$-\frac{E_0(Z_T)}{P_0(B_T)}(1+\rho)^T\omega^2. \tag{A22}$$

NOTES

1. The debt covenants of leveraged buyouts or firms recapitalized under distress commonly forbid distributing excess cash to shareholders or reinvesting it in nonliquid assets. Excess cash can either be invested in marketable securities (at a rate well below the cost of the firm's debt) or, sometimes mandatorily, used to retire debt. In cases in which the cash flow is below the minimum amortization requirements, the firm may be able to use revolving credit facilities, issue new debt, or renegotiate the terms of existing debt.

2. That the cost of debt is not appropriate for discounting the tax shield when debt is contingent on the realization of the firm's cash flows was recognized by Myers (1974) for the case in which the firm maintains a constant leverage ratio. Miles and Ezzell (1980) solved the valuation problem under a constant leverage ratio.

3. Miles and Ezzell showed that w is the weighted average cost of capital as commonly defined.

4. The terminal value $E_0(Z_{T+1})/(w-g)$ is discounted back to the present at ρ, not at w. This is so because the terminal value is perfectly correlated with the unlevered cash flow Z_{T+1}. Discounting the terminal value at w is correct only when the firm maintains a constant leverage ratio from time zero onward.

5. As suggested by Inselbag and Kaufold (1989). See also Kaplan and Rubback (1995), who call this approach "compressed APV."

6. The linkage between ρ and the levered cost of equity for these cases is provided by the beta-leveraging formulas of Hamada (1972) and Miles and Ezzell (1980).

7. See Arzac (1992), for example, for the role of equity kickers in highly leveraged transactions.

8. The tax shield cash flow is not proportional to the debt cash

flow when debt changes over time. Even if the risk characteristics of both sets of cash flow were still the same, discounting the tax shield at the promised yield to maturity of a term loan may not result in the correct value. Schaefer (1977) discusses the limitations of the promised yield to maturity.

9. Other contingent claim models of the leveraged firm proposed in the literature include those of Geske (1977); Brennan and Schwartz (1978); Turnbull (1979); Kane, Marcus, and McDonald (1985); and Toft (1994). These authors were not concerned with the case of uncertain leverage.

10. Dividends δY_t can be added here as in the APV model. Then, bondholders receive $\{(1-\delta)Y_1, \ldots, (1-\delta)Y_T + \min[B_T, V_T]\}$, $B_0 = (1-\delta)P_{0,T} + P_0(V_T) - c(V_T, B_T, T)$ and $S_0 = \delta P_{0,T} + c(V_T, B_T, T)$, where $\delta P_{0,T} = \delta(1-\delta)^{-1}[B_0 - P_0(V_T) + c(V_T, B_T, T)]$.

11. Allowing for the possibility of premature expiration of the equityholders' call would require valuing equity as a compound option in which minimum amortization and interest payments have to be made at each period in order to keep the next period option alive. See Geske (1977).

12. The Fisher–Margrabe formula holds when the constant volatility is replaced by the average volatility over the remaining life of the option.

13. It is straightforward to make dividends a distributed lag of Y_t, in order to allow for the smoothing of dividends observed by Lintner (1956) and Fama and Babiak (1968).

14. Note that discounting is done at r_f because $P_0(B_T)$ is the value of debt expected to be outstanding at time T if the option is exercised and no default occurs.

15. μ_2 and σ_2 are the *average* moments over $(0,T)$ because the $P(B_T)$ process has time-varying (nonstochastic) drift and variance rates.

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