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The Leverage Structure of Interest Rates

WHILE IT SEEMS INTUITIVELY CLEAR that competitive credit markets should generate an increasing leverage structure of interest rates, and while Fisher's [7] classic study of bond yields gives empirical evidence of a significant leverage premium, the theoretical literature contains only fragmentary development of conditions for a leverage structure. The purpose of this paper is to present sufficient lender supply and borrower demand conditions for a leverage structure to characterize equilibrium in competitive markets. The model allows for default, and assumes that lenders and borrowers are price takers, that they maximize concave utility functions, that they face an institutional regime of limited liability, and that they can invest both in a competitively available risk asset yielding stochastic constant returns and in a risk-free asset. Our framework is more general than mean-variance analysis; the latter would not be desirable in any event due to its inability to deal with a non-zero default probability under general utility assumptions.¹

¹Some of the early mean-variance literature may be read to imply a leverage structure of interest rates, but only under very restrictive utility assumptions. Tobin's [17] result that a risk averse investor demands a higher interest rate on consols as the variance of capital gains rises can be extended directly to loans having positive default probability (and hence a truncated distribution) only by assuming quadratic utility. Mossin [12] allows for default, but only in the special case of an independent two-point distribution of terminal wealth, and he implicitly assumes the case of quadratic utility plus the CAPM assumption of riskless lending and borrowing. His utility assumption cannot be relaxed by alternatively assuming, e.g., a normal distribution, because of the truncation of the lender's returns distribution induced by the possibility of default.

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Earlier works that have in passing stated supply side conditions for an increasing leverage structure are an example of Boness [6] and Barro's [5] elegant model of interest rate determination on collateral loans in the presence of default costs. Both assume lender risk neutrality (which yields the special case result that the lender's expected return on a leveraged loan is constant with respect to the leverage ratio, although the contractual interest rate rises) and neither states demand side conditions for support of a leverage structure. Baltensperger [4] and Stapleton [16] assume a positive leverage structure of interest rates in their contributions to the literature of credit rationing and the leverage structure of the firm, respectively. A leverage structure of interest rates is particularly important for the issue of credit rationing since it would substitute price rationing for quantity rationing and would be Pareto efficient [see the Smith [15] and Baltensperger [4] exchange]. However, since institutional constraints on interest rates commonly exist, the seminal Jaffee and Modigliani [9] contribution is a natural approach to the study of the effects of regulation or imperfect competition in credit markets. We consider lender behavior under a flat structure in another paper [3].

Section 1 of this paper presents a model of the credit transaction between individuals embedded in a competitive credit market. Section 2 obtains conditions for the optimal supply of leverage by price-taking lenders and derives restrictions on the leverage structure of the contractual interest rate and the risk premium. Section 3 shows that differences in borrower wealth or risk aversion are sufficient conditions for borrower support of an increasing leverage structure. Section 4 then presents an arbitrage proof of the necessity of an increasing leverage structure for competitive credit market equilibrium; we do not, however, undertake the considerably more formidable task of developing a proof of the existence of equilibrium under a leverage structure of interest rates. Section 5 contains our conclusions.

1. A MODEL OF COMPETITIVE CREDIT BEHAVIOR UNDER UNCERTAINTY

This section presents a model of the credit transaction between individuals embedded in a competitive credit market. In this model, the individual has to decide on the optimal allocation of his initial investible capital K , among a risk asset with random rate of return $R-1$ available to all investors, a riskless asset with return $s-1$, and lending to or borrowing from another individual. Short sales of the risk asset are not permitted.²

The amount of the loan is denoted by $|L|$, with $L > 0$ denoting a lender and $L < 0$ denoting a borrower. Individuals who borrow an amount $|L|$ to invest in the risk asset put up their entire investment in the risk asset as collateral. The borrower has limited liability in that if he defaults, he retains the portion of his capital M invested in the riskless asset; the remainder of his capital, $C (=K-M)$, is at risk.

²We note that all the results of this paper hold with trivial modifications if the individual also receives an exogenously-determined random "wage" income. Since inclusion of this variable would add no insight into credit behavior, it has been deleted to simplify the exposition.

The quality of the loan is measured by the leverage ratio

$$V = \frac{|L|}{C + |L|} .$$

Investors are assumed to be price takers, that is, to take as given whatever contractual interest rate, $r(V) - 1$, may be determined in the market.³ Like any price takers, lenders can choose the size (L) and the quality (V) of the loan, but must take the corresponding market price. Under limited liability, the random interest rate on the loan is

$$\tilde{r} = \begin{cases} r & \text{if } R \geq R^* (= rV) \\ R/V & \text{if } R < R^* . \end{cases}$$

R^* is the default rate of return on the risk asset; any return below R^* is insufficient to repay the loan in full.

With this notation and our assumptions, the individual's terminal wealth can be written

$$W = (K - M - L)R + L\tilde{r} + Ms .$$

Assuming the investor has a differentiable subjective probability distribution of returns on the risk asset $F(R)$, with density $f(R)$ and $F(0) = 0$,⁴ and a twice differentiable utility function $U(W)$, his expected utility may be written

$$\begin{aligned} E(U) &= \int_0^{R^*} U[(K - M + C)R + Ms]dF(R) \\ &\quad + \int_{R^*}^{\infty} U[(K - M - L)R + Lr + Ms]dF(R), \\ L &\geq 0 \text{ (lender),} \end{aligned} \tag{1}$$

$$\begin{aligned} E(U) &= U(Ms)F(R^*) + \int_{R^*}^{\infty} U[(K - M - L)R + Lr + Ms]dF(R), \\ L &< 0 \text{ (borrower) .} \end{aligned} \tag{2}$$

Any given individual can decide to be a lender or a borrower depending on his preferences and probability beliefs. The first order conditions for optimal lending

³We take r to be a differentiable function of V ; sufficient conditions for $r'(V) > 0$ will be shown later, but we make no assumption as to the slope of $r(V)$ at this point.

⁴The probability beliefs of borrowers and lenders are not assumed to be homogeneous in this paper. However, we do not deal with the implications of heterogeneity. If individuals take informational asymmetry into account, their optima as well as the resulting market equilibrium are likely to be affected as a consequence. See [10] and [11].

and investment are derived by differentiating (1) with respect to L , M , and C using Leibniz' Rule:⁵

$$\int_{R^*}^{\infty} U'[(K - M - L)R + Lr + Ms] \left(r + L \frac{\partial r}{\partial L} - R \right) dF(R) = 0, \quad (3)$$

$$\int_0^{\infty} U'(W) (s - R) dF(R) = 0, \quad (4)$$

$$\begin{aligned} & \int_0^{R^*} U'[(K - M + C)R + Ms] R dF(R) \\ & + \int_{R^*}^{\infty} U'[(K - M - L)R + Lr + Ms] L \frac{\partial r}{\partial C} dF(R) = 0. \end{aligned} \quad (5)$$

The conditions for optimal borrowing and investment follow from differentiating (2) with respect to L and M . The condition for optimal L is formally identical to (3).⁶ The condition for optimal M is now

$$\begin{aligned} U'(Ms)sF(R^*) + \int_{R^*}^{\infty} U'[(K - M - L)R + Lr + Ms] \\ \left(s - R + L \frac{\partial r}{\partial M} \right) dF(R) = 0. \end{aligned} \quad (6)$$

Example: This simple example illustrates how the model can generate borrowing and lending solutions. Assume that no riskless asset is available for investment (i.e., $M = 0$ above) and let $U(W) = -\exp(cW)$, $c < 0$, and $f(R) = \alpha \exp(-\alpha R)$, $\alpha > 0$, for $R \geq 0$. Substituting U and f into (3) yields

$$L = K + \frac{1}{c} \left[\frac{1}{(1 - V)r + L \frac{\partial r}{\partial L}} - \frac{1}{E(R)} \right],$$

which gives $L < 0$ for sufficiently small $|c|$ (absolute risk aversion) and r , and sufficiently large $E(R)$, and $L > 0$ vice versa.

Some Properties of the Optimum

Condition (3), which applies to both lending and borrowing, can be written

$$\frac{E_{R^*}^{\infty}[U'(W)R]}{E_{R^*}^{\infty}[U'(W)]} = r + L \frac{\partial r}{\partial L}, \quad (3')$$

⁵In deriving (3), we have made use of the following equality:

$$(K - M + C)R^* = (K - M - L)R^* + Lr, \quad \text{for } L > 0.$$

A statement of Leibniz' rule may be found in [1, p. 220], for example. Note that by taking the derivative with respect to L and C (the borrower's equity in the risk asset) the lender effectively chooses among borrowers so as to obtain optimal L and V . In the borrower's decision, below, choice of L and M determines C .

⁶In deriving (3) for borrowing, note that $(K - M - L)R^* + Lr = 0$ when $L < 0$.

where $E_{R^*}^\infty$ is the partial expectation operator over $[R^*, \infty)$. That is, at the optimum, the return on the risk asset weighted by the marginal utility of terminal wealth over $[R^*, \infty)$ should equal the marginal gross interest rate on the loan.

From (4) we obtain

$$\frac{E[U'(W)R]}{E[U'(W)]} = s, \quad (4')$$

which is the well-known condition for the optimal portfolio of a risky and a riskless asset [2, pp. 99–100]. In the next section we show that, as one might expect, (3') and (4') are intimately related.⁷

One can show (see app. A) that conditions (3)–(6) satisfy the first-order conditions for a Pareto efficient allocation of claims between a borrower and a lender constrained only by the structure of returns (including the leverage structure) and by limited liability.

The Pareto efficiency of credit markets operating under a leverage structure was demonstrated by Baltensperger [4] for a more restrictive model. Earlier, Smith [15] found that, under a flat structure, the borrower's capital acts as an external economy to the borrower. The next section formalizes the notion, put forward in various ways by Boness [6], Baltensperger [4], and Barro [5], that lenders will require higher interest rates in order to supply higher leverage loans.

2. THE SUPPLY OF LEVERAGE

Examination of (5) reveals that when $Pr[R \geq R^*] = 1$ (zero probability of default), the first term equals zero and the condition requires (for $L > 0$)

$$\frac{dr}{dC} = -\frac{V^2}{L} r'(V) = 0, \quad \text{or } r'(V) = \frac{dr}{dV} = 0.$$

Furthermore, the left-hand sides of (3') and (4') are identical and the two conditions can be simultaneously satisfied only if

$$r = s, \text{ for } Pr[R \geq R^*] = 1.$$

That is, the interest rate on a default-free loan should equal the riskless interest rate.

More interesting is the case where $Pr[R \geq R^*] < 1$ (positive probability of default). In this case (5) implies

$$r'(V) = \frac{\int_{R^*}^{R^*} U'(W)RdF(R)}{V^2 \int_{R^*}^{\infty} U'(W)dF(R)} > 0, \quad (7)$$

⁷Condition (6) also leads to a similar expression, but the random return on the risk asset appearing in the numerator now is s for $R \in [0, R^*)$ and $R - L \partial r / \partial M$ for $R \in [R^*, \infty)$.

since $L \partial r / \partial C = -V^2 r'(V)$. Hence we can state:

PROPOSITION 1. *A competitive lender's leverage supply function must be such that:*

$$\begin{aligned} r'(V) &= 0, & r &= s, \text{ for } V \text{ such that } \Pr[R \geq R^*] = 1, \\ r'(V) &> 0, & r &> s, \text{ for } V \text{ such that } \Pr[R \geq R^*] < 1. \end{aligned}$$

Note that (7) and proposition 1 do not depend on the sign of U'' and are, therefore, valid also for risk indifference or risk seeking. Moreover, they do not depend on the assumption that lenders can invest in the same risk asset as borrowers. Denoting by Q the return on the asset available to the lender, the following trivial changes apply to (7): $F = F(R, Q)$, and the domains of integration in numerator and denominator become $[0, R^*] \times [0, \infty)$ and $[R^*, \infty) \times [0, \infty)$, respectively. In particular, no risk asset need be available to the lender.

Barro [5] derives a result equivalent to our (7) for the case of a risk neutral lender unable to invest in the risk asset (but with the additional feature of explicit default costs).⁸ The Boness and Barro assumption of lender risk neutrality implies that the lender's *expected* rate of interest on the loan equals the risk-free rate and is therefore constant with respect to leverage. This is, of course, the case with our model. Note that

$$E(\tilde{r}) = \int_0^{R^*} \frac{R}{V} dF(R) + \int_{R^*}^{\infty} r dF(R)$$

and

$$\frac{d}{dV} E(\tilde{r}) = - \int_0^{R^*} \frac{R}{V^2} dF(R) + \int_{R^*}^{\infty} r' dF(R),$$

which is precisely the form taken by condition (5) for the lender's optimum under risk neutrality and, therefore, equals zero.

Under lender risk aversion, it can be shown that the *expected* rate of interest on the loan rises with leverage. In order to establish this result we first prove the following:

LEMMA 1. *Consider the functions $h(R)$, $k(R)$, and $F(R)$ defined on $[R_1, R_2]$ such that*

$$\begin{aligned} h(R) &\geq 0 \text{ as } R \geq R^{\circ}, \int_{R_1}^{R_2} h(R) dF(R) = 0, k(R) \text{ is positive and monotonic and} \\ F(R) &\text{ is positive and increasing on } [R_1, R_2]. \text{ Then } \operatorname{sgn} \int_{R_1}^{R_2} k(R) h(R) dF(R) = \\ &\operatorname{sgn}[k(R_2) - k(R_1)]. \end{aligned}$$

Proof.

$$\int_{R_1}^{R_2} k(R) h(R) dF(R) \geq \int_{R_1}^{R^{\circ}} k(R^{\circ}) h(R) dF(R) + \int_{R^{\circ}}^{R_2} k(R^{\circ}) h(R) dF(R)$$

⁸See Barro [5, eq. (14), p. 449].

$$\left[=k(R^0) \int_{R_1}^{R_2} h(R) dF(R) = 0 \right] \text{ as } k(R_2) - k(R_1) \geq 0 . \quad \text{Q.E.D.}$$

Let us define

$$\theta = \begin{cases} -R/V^2, & \text{for } R \in [0, R^*) \\ r', & \text{for } R \in [R^*, \infty); \end{cases}$$

hence we can write

$$\frac{d}{dV} E(\tilde{r}) = E(\Theta) .$$

Likewise, condition (5) becomes

$$E(U'\theta) = 0 . \quad (5')$$

Risk neutrality gives $E(\theta) = 0$ as stated above. In order to sign $E(\theta)$ under risk aversion ($U' > 0$, $U'' < 0$), we apply lemma 1 with $k(R) = 1/U'$ and $h(R) = U'\theta$, and write $E(\theta) = E(hk)$. Note that $k(R)$ is continuous and $k'(R) = -(1/U'')$ ($dW/dR > 0$ for $R \neq R^*$, hence $k(R)$ is increasing. Moreover, $h(R) \geq 0$ as $R \geq R^*$, and from (5'), $E(h) = E(U'\theta) = 0$. Applying lemma 1 we conclude that $E(\theta) = E(hk) > 0$. Hence, we have proved the following:

PROPOSITION 2. *Under risk aversion the lender's expected rate of interest on the loan is an increasing function of leverage.*

This means that the risk premium $E(\tilde{r}) - s$ increases with leverage under risk aversion. If that were not the case, and $E(\theta) = 0$, lemma 1 then implies that $E(U'\theta) < 0$ for $U'' < 0$, which, in the terminology introduced by Rothschild and Stiglitz [14], means that higher leverage would produce a mean-preserving increase in risk rejected by every risk averter.

3. THE DEMAND FOR LEVERAGE

Turning to the demand side, it is not obvious that borrowers will support an increasing leverage structure by demanding loans at more than one point on the structure. In this section we show that differences in wealth and degree of risk aversion are alternative sufficient conditions to produce demand for loans of different leverage in the presence of an increasing leverage structure of interest rates.

Response to Change in Wealth

In order to study the demand for leverage as a function of wealth, we need to derive the comparative statics responses of the optimal solution for the borrower.

Given the form of the expected utility function, the analysis is cumbersome if we consider simultaneously the responses of L and M to a change in wealth. Hence, we shall consider the simpler case where the investment in the riskless asset is not subject to decision, specifically where there is an equal proportional change in the two forms of wealth, that is:⁹

$$dK = dC + dM, \quad \text{where } dM = (M/C) dC. \quad (8)$$

Since no decision concerning M has to be made, only condition (3) applies.

We now prove:

PROPOSITION 3. *For dC as in (8), the demand for loans satisfies*

$$\text{sgn} \frac{dV}{dK} = - \text{sgn} \frac{d\rho_R}{dW},$$

where $\rho_R = -U''(W)W/U'(W)$ is the Arrow [2] and Pratt [13] relative risk aversion measure.

Proposition 3 is claimed for any arbitrary (market determined) structure $r(V)$, and in particular for $r'(V) > 0$. We first note that (see app. B)

$$\text{sgn} \frac{dV}{dK} = - \text{sgn} C \frac{\partial^2 E(U)}{\partial C \partial L} + L \frac{\partial^2 E(U)}{\partial L^2}, \quad (9)$$

where $C = K - M$.

Furthermore, it is shown in Appendix (C) that taking the derivatives of (3) with respect to K and L and substituting into the right hand side of (9) yields

$$\text{sgn} \frac{dV}{dK} = - \text{sgn} \int_{R^*}^{\infty} \rho(W) U'(W) \left[R - \left(R + L \frac{\partial r}{\partial L} \right) \right] dF(R). \quad (10)$$

Finally, letting

$$\rho(W) = k(R), \text{ and } U'(W) \left[R - \left(r + L \frac{\partial r}{\partial L} \right) \right] = h(R)$$

in (10), and applying lemma 1, completes the proof of proposition 3.

We have thus established that for the case of an equal proportional change in wealth invested in risky and riskless assets, the demand for higher leverage loans decreases (increases) with wealth for increasing (decreasing) relative risk aversion. Hence, since proposition 3 holds for $r'(V) > 0$, and since borrowers generally differ in wealth levels, some borrowers will be willing to pay the higher r required by lenders in order to obtain higher leverage.

⁹A more general case is considered in app. D.

Proposition 3 can be expressed in terms of the wealth elasticity of the demand for loans. Let $\eta = (K/L) (dL/dK) = (C/L) (dL/dC)$. Then

$$\frac{dV}{dC} = - \frac{L}{(C-L)^2} (\eta - 1)$$

(see exp. (A8) in app. B). Hence, since $L < 0$ and $dV/dK = (C/K)dV/dC$, we have the following:

COROLLARY. *For dC as in (8), the demand for loans satisfies $\text{sgn} (\eta - 1) = - \text{sgn} \frac{d\rho}{dW}$.*

It is trivial to verify that proposition 3 also holds in the particular case in which no riskless asset is available and $M = 0$.

Response to Changes in Attitude Toward Risk

We now consider the effect on L and V of changes in the borrower's attitude toward risk under the assumption of predetermined investment in the riskless asset. Consider the case of constant nonunitary relative risk aversion utilities:

$$U(W) = \frac{1}{\gamma} W^\gamma, \quad \gamma \neq 0, < 1, \quad (11)$$

where $\gamma = 1 - \rho_R(W)$. Differentiating (3) (for $L < 0$) totally with respect to γ gives the borrower's optimal response to a decrease in the degree of relative risk aversion:

$$\text{sgn} \frac{dL}{d\gamma} = \text{sgn} \frac{E(U)}{\partial \gamma \partial L}.$$

Substituting (11) into (2) and differentiating with respect to L and γ one obtains

$$\frac{\partial^2 E(U)}{\partial \gamma \partial L} = - \int_{R^*}^{\infty} (\log W) W^{\gamma-1} \left[R - \left(r + L \frac{\partial r}{\partial L} \right) \right] dF(R). \quad (12)$$

Finally, letting $\log W = k(R)$ and $W^{\gamma-1} (r + L(\partial r/\partial L) - R) = h(R)$ in (12), and applying lemma 1, gives $dL/d\gamma < 0$.

A similar result can be established for the class of constant absolute risk aversion utilities $U(W) = - \exp(cW)$, $c < 0$. Therefore, recalling that $L < 0$ for the borrower, we have proved:

PROPOSITION 4. *Borrowers with different constant relative or absolute risk aversion utilities will, ceteris paribus, demand higher leverage loans the lower their degree of risk aversion, even when the leverage structure of interest rates is increasing.*

Propositions 3 and 4 constitute a set of reasonably general diversity conditions sufficient to guarantee that borrowers will demand loans of different leverage in the presence of an increasing leverage structure of interest rates.

4. AN ARBITRAGE PROOF THAT COMPETITIVE EQUILIBRIUM IS CHARACTERIZED BY AN INCREASING LEVERAGE STRUCTURE OF INTEREST RATES

The results we have shown thus far apply to individual lenders and borrowers. These results do not immediately imply that across all lenders and loan sizes a nonincreasing leverage structure could not characterize equilibrium. We now prove the more powerful result:

PROPOSITION 5. *A nonincreasing leverage structure of interest rates is incompatible with equilibrium in a competitive credit market in the sense that it would permit the realization of pure arbitrage profits.*

To prove this proposition we have to extend the scope of our model to allow for the possibility of financial intermediaries. The proof will consist of showing that a nonincreasing leverage structure would enable intermediaries to realize pure arbitrage profits.

Proof. Let $r'(V) \leq 0$ and consider an intermediary who borrows L_I , offering the lender collateral $L_I + C_I$, and lends a borrower $L_B = L_I$, obtaining collateral $L_B + C_B > L_I + C_I$. That is, the intermediary uses only part of the collateral pledged by the borrower to cover his loan, and hence $V_B < V_I$. With $r'(V) \leq 0$, we have $r(V_B) \geq r(V_I)$ so the intermediary need pay no interest rate penalty for withholding part of the borrower's collateral from his pledge to the lender. There are three distinct ranges of R (the return on the underlying risk asset) to be considered. Each range with its corresponding default characterization and expression for final wealth is given in Table 1. The first inequality in the last column of this table follows from $C_B > C_I$. The second inequality follows from the default conditions $R < r(V_I)V_I$ and from $r'(V) \leq 0$, which imply $R < r(V_B)V_I = r(V_B)L_I/(C_I + L_I)$, or $L_I r(V_B) > (C_I + L_I)R$. Finally, the third inequality follows from $r'(V) \leq 0$.

Intermediation increases wealth over $[0, rV_I)$ and increases (for $r'(V) < 0$) or maintains (for $r'(V) = 0$) wealth over $[rV_I, \infty)$. Hence, the intermediary has succeeded in constructing a probability distribution of terminal wealth that absolutely dominates the previous distribution for any increasing utility function [8], and interme-

TABLE 1

ARBITRAGE—INTERMEDIATION OUTCOMES

Range of Risk Asset Return	Loans Defaulted	Final Wealth Generated by Intermediation
$[0, r(V_B)V_B)$	L_I, L_B	$(C_B - C_I)R > 0$
$[r(V_B)V_B, r(V_I)V_I)$	L_I	$L_I r(V_B) - (C_I + L_I)R > 0$
$[r(V_I)V_I, \infty)$	none	$L_I[r(V_I) - r(V_I)] \geq 0$

diation is unambiguously profitable. A similar argument rules out the possibility of loans of different size following a nonincreasing leverage structure.¹⁰

In attempting to induce borrowers to accept more low leverage loans, intermediaries would bid down the interest rate on such loans. So long as the supply curve of loans is not so backward bending as to drive interest rates for high loans down faster, intermediation will not have an explosive impact and will produce a positive leverage structure across all V .¹¹ In other words, if a competitive equilibrium exists, it must be characterized by a positive leverage structure of interest rates.

5. CONCLUSION

The focus of this paper has been the effect of leverage on interest rates. While we have not attempted to state conditions for the existence of a competitive equilibrium characterized by a leverage structure of interest rates, we have shown that if a competitive equilibrium exists, it will exhibit an increasing leverage structure of interest rates.

On the supply side, positive marginal utility of wealth is the only restriction required to make an increasing leverage structure necessary for an interior maximum of the lender's utility function. On the demand side, risk averse borrowers will support an increasing leverage structure by demanding loans at more than one point on the structure if, besides some other mild assumptions, there are differences in wealth among borrowers or differences in the degree of absolute or relative risk aversion.

By allowing for the possibility of arbitrage intermediation, we also derive an arbitrage proof that rules out a nonincreasing leverage structure in a competitive market.

No statement of demand side conditions for support of an increasing leverage structure by borrowers appears in the literature, and Barro's [5] supply side conditions are stated for the special case of a risk neutral lender unable to invest in the risk asset (although, as we see here, they may be extended a fortiori to risk aversion). To our knowledge, the arbitrage proof and its use of stochastic dominance are also new. We have also proved that the risk premium on loans (i.e., the difference between the *expected* interest rate and the riskless rate) will be an increasing function of leverage.

Our work, taken together with the Baltensperger [4] and Barro [5] contributions, serves to clarify the relationship between the leverage structure of interest rates and

¹⁰Suppose $r(V_1) \leq r(V_2)$ for some $V_1 > V_2$ when $L_1 > L_2$. Then the intermediary can realize pure arbitrage profits by borrowing L_1 with leverage V_1 and making two loans: L_2 and $L_3 (= L_1 - L_2)$, each with leverage $V_2 = V_3$. We note that $C_2 + C_3 > C_1$. That is, the collaterals pledged to the intermediary exceed his collateral pledge.

¹¹One can show the present model admits a backward bending supply of loans. See Jaffee and Modigliani [9] and Smith [15] for detailed analyses of this property in models that are specific instances of ours. Explosive intermediation would occur only with a negatively inclined supply curve that never intersects demand from above as loan quantity increases. Demonstration of the existence and stability of equilibrium are beyond the scope of this paper.

credit rationing: when a competitive equilibrium is allowed to and does occur, it will (at least under the conditions we have assumed here) be characterized by an increasing leverage structure of interest rates, and there will not be credit rationing in the accepted meaning (i.e., there will be price, instead of quantity rationing). When a flat leverage structure is imposed, leverage will be rationed as in the models of Jaffe and Modigliani [9] and Smith [15].

APPENDIX

A. Pareto Efficiency

We shall establish that conditions (3)-(6) satisfy the first-order conditions for a Pareto efficient allocation of claims. Let X_1 and $C + l$ denote the lender's and the borrower's investment in the risk asset, respectively (l denotes the amount of the loan). The Lagrangian of the Pareto optimization problem then is

$$\begin{aligned} \phi = & \int_0^{R^*} U_1 [(X_1 + C + l)R + M_1 s] dF_1(R) + \int_{R^*}^{\infty} U_1 [X_1 R + l r + M_1 s] \\ & dF_1(R) + \lambda \left[U_2(M_2 s) F_2(R^*) + \int_{R^*}^{\infty} U_2 [(C + l)R - l r + M_2 s] \right. \\ & \left. dF_2(R) + \mu (K_1 + K_2 - X_1 - C - l - M_1 - M_2), \right. \end{aligned}$$

where λ is an arbitrary welfare weight and μ is the multiplier of the planner's budget constraint.

Differentiating ϕ with respect to X_1 , M_1 , l , C , and M_2 , respectively, yields

$$\int_0^{\infty} U_1' R dF_1(R) - \mu = 0, \quad (\text{A1})$$

$$\int_0^{\infty} U_1' s dF_1(R) - \mu = 0, \quad (\text{A2})$$

$$\begin{aligned} & \int_0^{R^*} U_1' R dF_1(R) + \int_{R^*}^{\infty} U_1' \left(r + l \frac{\partial r}{\partial L} \right) dF_1(R) + \\ & \lambda \int_{R^*}^{\infty} U_2' \left(R - r - l \frac{\partial r}{\partial L} \right) dF_2(R) - \mu = 0, \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} & \int_0^{R^*} U_1' R dF_1(R) + \int_{R^*}^{\infty} U_1' l \frac{\partial r}{\partial C} dF_1(R) + \\ & \lambda \int_{R^*}^{\infty} U_2' \left(R - l \frac{\partial r}{\partial C} \right) dF_2(R) - \mu = 0, \end{aligned} \quad (\text{A4})$$

$$U'_2(M_{2s})dF_2(R) + \int_{R^*}^{\infty} U'_2 s dF_2(R) - \frac{\mu}{\lambda} = 0. \quad (A5)$$

Eliminating μ from (A1) and (A2) gives (4). Eliminating μ from (A1) and (A3) yields

$$\int_{R^*}^{\infty} U'_1 \left(r + l \frac{\partial r}{\partial L} \right) dF_1(R) + \lambda \int_{R^*}^{\infty} U'_2 \left(R - r - l \frac{\partial r}{\partial L} \right) dF_2(R) = 0, \quad (A6)$$

which is satisfied by (3), since each of the first and second integrals of (A6) equals zero by condition (3) for the lender and the borrower, respectively. Finally, the sum of the first two integrals in (A4) equal zero by (5). Hence,

$$- \frac{\mu}{\lambda} = \int_{R^*}^{\infty} U'_2 \left(-R + l \frac{\partial r}{\partial C} - R \right) dF_2(R) = 0, \quad (A7)$$

which, upon substitution into (A5), yields (6). (Note that $l \partial r / \partial C$ in (A7) equals $L \partial r / \partial M$ in (6).)

B. Derivation of Expression (9)

Differentiating $V = -L/(C - L)$ yields

$$\frac{dV}{dC} = - \frac{dL}{dC} \frac{C}{(C - L)^2} + \frac{L}{(C - L)^2} \quad (A8)$$

but, from the total differential of (3) w.r.t. C one obtains

$$\frac{dL}{dC} = - \frac{\partial^2 E(U) / \partial C \partial L}{\partial^2 E(U) / \partial L^2}, \quad (A9)$$

where $\partial^2 E(U) / \partial L^2 < 0$ by the second-order condition. Substituting (A9) into (A8) yields

$$\frac{dV}{dC} = \frac{L}{(C - L)^2} \left[- \frac{C}{L} \frac{\partial^2 E(U) / \partial K \partial L}{\partial^2 E(U) / \partial L^2} + 1 \right],$$

which implies (9), since $L < 0$ and $dV/dK = (C/K)dV/dC$.

C. Derivation of Expression (10)

Substituting the derivatives of (3) w.r.t. C ($= K - M$) and L into the left-hand side of (9) yields

$$\begin{aligned}
 C \left[\int_{R^*}^{\infty} U''(W) \left(R + L \frac{\partial r}{\partial C} + s \frac{dM}{dC} \right) \left(r + L \frac{\partial r}{\partial L} - R \right) dF(R) \right. \\
 + \int_{R^*}^{\infty} U'(W) \left(\frac{\partial r}{\partial C} + L \frac{\partial^2 r}{\partial C \partial L} \right) dF(R) - U'(Ms) \\
 \left. \left(r + L \frac{\partial r}{\partial L} - R^* \right) f(R^*) \frac{\partial R^*}{\partial C} \right] + L \left[\int_{R^*}^{\infty} U''(W) \right. \\
 \left. \left(r + L \frac{\partial r}{\partial L} - R \right)^2 dF(R) + \int_{R^*}^{\infty} U'(W) \left(2 \frac{\partial r}{\partial L} + L \frac{\partial^2 r}{\partial L^2} \right) \right. \\
 \left. dF(R) - U'(Ms) \left(r + L \frac{\partial r}{\partial L} - R^* \right) f(R^*) \frac{\partial R^*}{\partial L} \right]. \quad (A10)
 \end{aligned}$$

Next, differentiation of $R^* = rV$, $r = r(V)$, $V = -L/(C - L)$ shows that

$$C \frac{\partial Z}{\partial C} + L \frac{\partial Z}{\partial L} = 0, \quad \text{for } Z = V, R^*, r. \quad (A11)$$

Hence the terms involving $U'(Ms)$ cancel and one term involving $\partial r/\partial C$ and $\partial r/\partial L$ in each of the U'' and U' expression also cancels. Now, substituting $dM/dC = M/C$ from (8) and recognizing that $CR - LR + Lr + Ms = W$, (A10) becomes

$$\begin{aligned}
 \int_{R^*}^{\infty} U''(W) W \left(r + L \frac{\partial r}{\partial C} - R \right) dF(R) + \int_{R^*}^{\infty} U'(W) L \\
 \left(C \frac{\partial^2 r}{\partial C \partial L} + L \frac{\partial^2 r}{\partial L^2} + \frac{\partial r}{\partial L} \right) dF(R). \quad (A12)
 \end{aligned}$$

Finally, note that

$$\frac{\partial^2 r}{\partial C \partial L} = \frac{\partial}{\partial C} \left(\frac{dr}{dV} \frac{\partial V}{\partial L} \right) = \frac{d^2 r}{dV^2} \frac{\partial V}{\partial W} \frac{\partial V}{\partial L} + \frac{dr}{dV} \frac{\partial^2 V}{\partial C \partial L},$$

and

$$\frac{\partial^2 r}{\partial L^2} = \frac{d^2 r}{dV^2} \left(\frac{\partial V}{\partial L} \right)^2 + \frac{dr}{dV} \frac{\partial^2 V}{\partial L^2}.$$

Then, by (A11)

$$C \frac{\partial^2 r}{\partial C \partial L} + L \frac{\partial^2 r}{\partial L^2} = \frac{dr}{dV} \left(C \frac{\partial^2 V}{\partial C \partial L} + L \frac{\partial^2 V}{\partial L^2} \right) = - \frac{C}{(C - L)^2} \frac{dr}{dV}. \quad (A13)$$

But, since

$$\frac{\partial r}{\partial L} = \frac{dr}{dV} \frac{C}{(C - L)^2}, \quad (\text{A14})$$

(A13) equals $-\partial r/\partial L$ and the second term in (A12) equals zero. Multiplying the integral of the remaining term in (A12) by $U'(W)/U'(W)$ gives the right-hand side of (10).

D. Generalization of Proposition 3

We now derive dV/dK , where (8) is replaced by the milder assumption $dC/dK > 0$. That is, we assume that some of the increase in wealth is allocated to the risk asset. Note that Arrow's proposition [2, app. to essay 3] (that favorable risk assets exhibiting stochastic constant returns to scale are superior under decreasing absolute risk aversion) does not permit us to conclude that $dC/dK > 0$. C is only a fraction of the individual's total investment in the risk asset, and borrowing under limited liability destroys the stochastic constant returns to scale property of the risk asset.

The following additional notation is needed: $\psi = \rho W$ for the Arrow and Pratt absolute risk aversion measure, and $\chi = (C/M) dM/dC$ for the elasticity of investment in the riskless asset with respect to the individual's equity in the risk asset. We now prove:

PROPOSITION 3'.

Decreasing absolute risk aversion and $dC/dK > 0$ imply

$$\begin{aligned} \frac{dV}{dK} < 0, & \quad \text{if } \frac{d\rho}{dW} \geq 0, \text{ and } \chi < 1 \\ \frac{dV}{dK} \geq 0, & \quad \text{as } \frac{d\rho}{dW} \geq 0, \text{ if } \chi = 1 \\ \frac{dV}{dK} > 0, & \quad \text{if } \frac{d\rho}{dW} \leq 0, \text{ and } \chi > 0. \end{aligned}$$

dV/dK is undefined, otherwise.

Proof. Proceeding as in (B) we obtain

$$\text{sgn} \frac{dV}{dK} = - \text{sgn} \left(C \frac{\partial^2 E(U)}{\partial C \partial L} + L \frac{\partial^2 E(U)}{\partial L^2} + C \frac{\partial^2 E(U)}{\partial M \partial L} \frac{dM}{dC} \right) \frac{dC}{dK}. \quad (\text{A15})$$

Substituting the derivatives of (3) w.r.t. C, L , and M into the left-hand side of (A15), and using (A11), (A13), and (A14) as in (C) yields

$$\int_{R^*}^{\infty} U''(W)(CR + Lr) \left(r + L \frac{\partial r}{\partial L} - R \right) dF(R)$$

$$+ C \frac{dM}{dC} s \int_{R^*}^{\infty} U''(W) \left(r + L \frac{\partial r}{\partial L} - R \right) dF(R) .$$

Finally, multiplying the second term of this expression by M/M , adding and subtracting

$$Ms \int_{R^*}^{\infty} U''(W) \left(r + L \frac{\partial r}{\partial L} - R \right) dF(R),$$

and taking into account the definitions of W , ρ , ψ , and χ yields

$$\begin{aligned} & \int_{R^*}^{\infty} \rho(W) U'(W) \left[R - \left(r + L \frac{\partial r}{\partial L} \right) \right] dF(R) \\ & + (\chi - 1) Ms \int_{R^*}^{\infty} \rho(W) U'(W) \left[R - \left(r + L \frac{\partial r}{\partial L} \right) \right] dF(R) . \end{aligned} \quad (A16)$$

Applying lemma 1 to each term of (A16) gives proposition 3'.

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