

PROFITS AND SAFETY IN THE THEORY OF THE FIRM UNDER PRICE UNCERTAINTY*

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1. INTRODUCTION

THIS PAPER studies quantity-setting behavior under price uncertainty when preferences are based upon expected profits and the probability of loss. These preferences have been suggested over the years by several authors [1], [5], [6], [9], [10], [13] as alternatives to the abstract notion of a utility function and are not generally compatible with the axioms of utility theory.¹ Previous studies on the subject have been based on the restrictive assumption of normality or on Tchebychev's inequality.² This paper derives exact distribution-free results by the straightforward use of the monotonicity property of distributions. Section 2 presents the equilibrium conditions of competitive and monopolistic firms under alternative profit-safety preferences. It shows that the conclusions of Day, Aigner and Smith [5] are valid for arbitrary distributions under a rather general form of price uncertainty, which includes both additive and multiplicative uncertainty as special cases. Section 3 derives the comparative statics properties of the solutions for changes in fixed costs, price and taxes and shows that the response of optimal output depends critically on the form of the preference function. Section 4 compares the results of this paper with those of traditional theory and expected utility maximization.

2. OPTIMAL OUTPUT

This section derives the equilibrium conditions of competitive and monopolistic quantity-setting firms operating under price uncertainty and behaving according to one of the following criteria:

- (1) $\min P = Pr\{\pi \leq 0\},$
- (2) $\max E(\pi),$ subject to $Pr\{\pi \leq 0\} \leq \alpha,$
- (3) $\max z,$ subject to $Pr\{\pi \leq z\} \leq \alpha,$
- (4) $\max \Phi = E(\pi) + \varphi(P), \varphi'(P) < 0, \varphi''(P) < 0.$

The first three are the safety-first criteria proposed by Roy [10], Telser [13], and Kataoka [6], respectively. (1) requires setting output to minimize the prob-

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¹ On different and sometimes opposed opinions about the reasonability of utility theory axioms see Allais [1], Savage [12], Arrow [2], Thrall [14] and Chipman [4].

² Tchebychev's inequality gives an upper bound for the probability of loss. It results from the maximin criterion applied to an unknown distribution.

ability of loss.³ (2) involves maximizing expected profits under the constraint that the probability of loss does not exceed some critical level α . (3) requires maximizing the α -fractile of the profit distribution. (4) is a tractable specification of a criterion proposed by Allais [1] which admits substitution between expected profits and safety.⁴

Three of these criteria are not totally compatible with utility theory. It is easy to verify that (1) can be derived from a utility function [10] but that (2) and (3) are incompatible with the continuity and independence axioms and do not correspond to any real or vector-valued (lexicographic) utility [4]. Finally, (4) admits trade-off between expected profits and safety and, hence, satisfies continuity but only its linear form satisfies independence and is compatible with utility theory [1], [8].

The following notation and assumptions are introduced: let $p = p(x, u)$ be the random price, x the output, u a random disturbance with $E(u) = 1$, $c(x)$ the variable cost function and f the fixed costs. The demand function faced by the firm is of the form

$$(5) \quad p(x, u) = g(x) - h(x)u, \quad h(x) > 0,$$

which includes additive ($h(x) \equiv 1$) and multiplicative uncertainty ($g(x) \equiv 0$) as special cases. Expected price ($\bar{p} = E(p)$) is assumed to be nonincreasing in order to include both the competitive and imperfectly competitive cases, that is,

$$\frac{d\bar{p}}{dx} = g'(x) + h'(x) \leq 0.$$

Strictly, the competitive case does not require any assumption about the stochastic nature of p , but using (5) does not involve a loss of generality and avoids unnecessary repetitions. x_R , x_T , x_K and x_C will denote the solutions corresponding to criteria (1), (2), (3) and (4) and the quasi-certainty solution with $p = \bar{p}(x)$, respectively. The appropriate second order conditions are taken to hold in all cases.

The equilibrium conditions under each of the above criteria are now derived. Roy's criterion requires minimizing

$$Pr\{\pi \leq 0\} = Pr\left(u \leq \frac{AC - g}{h}\right),$$

where $AC = [c(x) + f]/x$. This is equivalent to minimizing $(AC - g)/h$ and, since

$$(6) \quad \frac{d[(AC - g)/h]}{dx} = \frac{(MC - \overline{MR})h - (AC - \bar{p})(h + xh')}{xh^2},$$

³ Loss is usually defined as including zero profits [5], [9], [10], but that is not strictly necessary.

⁴ Baumol's criterion [3] will not be considered in this paper. It can be interpreted as a generalization of Kataoka's and has similar axiomatic characteristics.

$$(7) \quad \overline{MR} = MC + \frac{1}{h}(\bar{p} - AC)(h + xh')$$

at the optimum, where $\overline{MR} = \bar{p} + x d\bar{p}/dx$, $MC = dc/dx$ and $h' = dh/dx$.

Since under additive uncertainty $h' = 0$, and under multiplicative uncertainty $\bar{p} = h$ and $MR = h + xh'$, (7) becomes

$$\overline{MR} = MC + (\bar{p} - AC) \quad \text{and} \quad \overline{MR} = MC \frac{\bar{p}}{AC},$$

respectively. Therefore, for positive expected profits,⁵ $\overline{MR} > MC$ and $x_R < x_C$, where x_C is the solution to $\overline{MR} = MC$. A sufficient condition for $x_R < x_C$ in the general case is that $h + xh' > 0$. This will be so if, as it is reasonable to expect and we assume henceforth, the dispersion of total revenue (σ_R) increases with output.⁶ In fact,

$$\frac{d\sigma_R}{dx} = (h + xh')\sigma_u,$$

where σ_u is the dispersion of the price disturbance.

Under competition $d\bar{p}/dx = g'(x) = h'(x) = 0$ and (7) becomes $AC = MC$, that is, Roy's criterion simply requires minimizing average costs.

According to Telser's criterion optimal output results from

$$(8) \quad \max E(\pi), \text{ subject to } Pr\left(u \leq \frac{AC - g}{h}\right) \leq \alpha,$$

which has solution $x_T = x_C$ if at x_C the constraint is satisfied. Otherwise,

$$\frac{AC(x_T) - g(x_T)}{h(x_T)} < \frac{AC(x_C) - g(x_C)}{h(x_C)},$$

and $x_T < x_C$, since (6) is positive for $E(\pi) > 0$ and $x > x_C$. Moreover, $x_T \geq x_R$ because $\min Pr\{\pi \leq 0\} \leq \alpha$ when (8) is feasible.

Kataoka's criterion becomes

$$\max z, \text{ subject to } Pr \left\{ u \leq \frac{1}{h} \left(\frac{z}{x} + AC - g \right) \right\} \leq \alpha.$$

Let Q be the α -fractile of u . Then, maximum z is attained at

$$(9) \quad \frac{1}{h} \left(\frac{z}{x} + AC - g \right) = Q, \quad \text{or} \quad z = (g + hQ)x - TC$$

where $TC = c(x) + f$. That is, z can be interpreted as the certainty equivalent profit. It has its maximum at x_K given by

⁵ If $E(\pi) = (\bar{p} - AC)x \leq 0$ the probability of loss will be intolerably high for plausible price distributions.

⁶ This is related but not equivalent to Leland's "principle of increasing uncertainty." For small risks this principle can be stated as $d\sigma_R/dR > 0$, which implies that $\text{sign}(d\sigma_R/dx) = \text{sign}(\overline{MR})$.

$$(10) \quad \overline{MR} = MC + (1 - Q)(h + xh'),$$

where $\overline{MR} > MC$ under the mild assumption that $Pr\{u \leq E(u)\} > \alpha$ which implies $Q < E(u) = 1$. Combining this condition with (9) leads to

$$\overline{MR} < MC + \frac{1}{h}(\bar{p} - AC)(h + xh')$$

which together with (7) shows that $x_K > x_R$. x_K and x_T cannot be compared in general, but it can be verified that $x_K < x_T$ for the same value of α .

The maximum of Allais' criterion is attained at the solution of

$$(11) \quad \frac{d\Phi}{dx} = \overline{MR} - MC + \phi's \left[\frac{(MC - \overline{MR})h - (AC - \bar{p})(h + xh')}{xh^2} \right] = 0,$$

where, for convenience, we have assumed that u has a density s . For positive expected profits (11) requires that $\overline{MR} > MC$ since otherwise $d\Phi/dx < 0$. Therefore, $x_A < x_C$. Under competition $\bar{p} > MC$ because the quasi-certainty long-run competitive equilibrium, which would result in $\bar{p} = MC = AC$ and satisfy (11), will not be attained. In fact, as $\bar{p}$ goes down toward AC , P approaches $Pr\{p \leq \bar{p}\}$ and discourages entry. Finally, $x_A > x_R$ because (11) will be positive at Roy's solution.

The general conclusion of this analysis is that *under price uncertainty criteria (1) to (4) result in competitive and monopolistic outputs that are usually smaller and never greater than their certainty counterparts*. This conclusion is similar to that obtained for risk averse utility maximizing firms by Leland [7] and Sandmo [11], and generalize the results obtained by Day, Aigner and Smith [5] for safety-first criteria and additive uncertainty using Tchebychev's inequality.

3. COMPARATIVE STATICS

3.1. *Changes in fixed costs.* Infinitesimal changes in fixed costs do not affect the equilibrium of the firm under certainty. Leland and Sandmo have shown that a utility maximizing firm with decreasing absolute risk aversion will decrease output when fixed costs increase. Each of the three safety-first criteria (1), (2) and (3) results in a different reaction. For a firm following Roy's criterion, differentiating (6) with respect to f yields

$$\frac{dx}{df} = \frac{h + xh'}{x^2h^2} \left\{ \frac{\partial^2[(AC - g)/h]}{\partial x^2} \right\}^{-1}$$

which is positive since $\partial^2[(AC - g)/h]/\partial x^2 > 0$ is the second-order condition for the optimum. Therefore, *Roy's firm will increase output when f increases*. If the constraint in (8) holds tightly and f increases, *Telser's firm will decrease output* to maintain feasibility since $\partial AC/\partial f > 0$ and $\partial[(AC - g)/h]/\partial x > 0$ at the optimum. *Fixed costs do not affect the output of Kataoka's firm* because its maximum condition (10) is independent of f . *Allais' firm may either increase or de-*

crease output as f increases depending on the characteristics of the profit-safety trade-off, demand and costs. In fact, differentiating (11) with respect to f we obtain

$$(12) \quad \frac{dx}{df} = -\frac{1}{x^2 h^2} \left(\frac{\partial^2 \Phi}{\partial x^2} \right)^{-1} \left[(\varphi'' s^2 + \varphi' s') x h \frac{d[(AC - g)/h]}{dx} - \varphi' s (h + x h') \right],$$

the sign of which cannot be established in general.⁷ A sufficient condition for output to decrease as f increases is

$$(13) \quad \frac{d \log \varphi'}{d \log \sigma_\pi} = \frac{\varphi'' x h}{\varphi' (h + x h')} \frac{dP}{dx} > 1$$

where $\sigma_\pi = x h \sigma_u$ and, as before, $P = Pr\{\pi \leq 0\}$. (13) implies that the expression in brackets in (12) is negative since it requires

$$x h \frac{d[(AC - g)/h]}{dx} = \frac{1}{s} x h \frac{dP}{dx} > \frac{\varphi'}{\varphi' s} (h + x h') > \frac{\varphi' s}{\varphi'' s^2 + \varphi' s'} (h + x h').$$

Therefore, output will fall if the marginal rate of substitution of safety for expected profits decreases more than proportionately to the induced decrease in the dispersion of profits.⁸

3.2. Increased price. Let us consider the effect of an upward shift in the demand curve:

$$p^+ = p + a = g(x) + h(x)u + a,$$

where a is the shift constant. Certainty output will increase when price increases. Similarly, the decreasing absolute risk averse—utility maximizing firm will increase output in response to an increase in price, but increasing risk aversion may result in output reduction [7].

The implications of Roy's criterion depend on the type of uncertainty. In fact, substituting p^+ in (6) and differentiating with respect to a yields

$$\frac{dx}{da} = -\frac{h'}{h} \left\{ \frac{\partial^2 [(AC - g)/h]}{\partial x^2} \right\}^{-1}$$

at $a = 0$, which shows that $\text{sign}(dx/da) = -\text{sign}(h')$. Therefore, when price increases the output of Roy's firm will not change if $h' = 0$ (additive uncertainty), will increase if $h' < 0$ (which includes multiplicative uncertainty) and will decrease if $h' > 0$.

⁷ $d[(AC - g)/h]/dx > 0$ and $\partial^2 \Phi / \partial x^2 < 0$ by the first and second-order maximum conditions. Also, it seems natural to expect that $\varphi'' s^2 + \varphi' s' < 0$, since $\varphi', \varphi'' < 0$ by assumption, and $s' > 0$ if price expectations are unimodal and the probability of loss becomes intolerably high right of the mode. We should point out that the second-order condition does not necessarily imply that $\varphi'' s^2 + \varphi' s' < 0$, nor results in other useful restrictions even in the cases of additive and multiplicative uncertainty.

⁸ φ' can be interpreted as the marginal rate of substitution of safety $(1 - P)$ for expected profits at a given level of preference $\Phi = \Phi^0$.

Increased price increases output in the case of criteria (2) and (3). An increase in price will relax the constraint in Telser's problem and allow an increase in output since $dE(\pi)/dx > 0$ when the constraint is tight. Since $\overline{MR}$ becomes $\bar{p} + x d\bar{p}/dx + a$, output must increase in order to satisfy Kataoka's maximum condition (10).

With a minor exception, *increased price results in a larger output under Allais' criterion.* Substituting p^+ in (11) and differentiating with respect to a we obtain

$$\frac{dx}{da} = - \left(\frac{\partial^2 \Phi}{\partial x^2} \right)^{-1} \left[1 - (\varphi'' s^2 + \varphi' s') \frac{1}{h} \frac{d[(AC - g)/h]}{dx} + \varphi' s \frac{h'}{h} \right]$$

at $a = 0$, which is positive for additive and multiplicative uncertainty by the maximum conditions and the fact that we expect $s' > 0$ at equilibrium.⁹ For the general form of (5), output will increase if $h' \leq 0$ but *might* decrease if $h' > 0$. It can be verified that

$$\frac{d \log \varphi'}{d \log \sigma_\pi} > 1 + \frac{h^2}{\varphi' s h'}$$

is sufficient for $dx/da > 0$. This condition is less stringent than (13) because $h^2/\varphi' s h' < 0$.

3.3. Response to profit and sales taxes. A proportional profit tax with full loss offset does not affect the output of the firm under certainty. Sandmo has shown that a competitive utility maximizing firm operating under price uncertainty will change its output in the same way as the Arrow-Pratt relative risk aversion measure when confronted with a tax rate increase. *The competitive or monopolistic firm behaving according to safety-first criterion (1), (2), or (3) will not change its output when the tax rate changes.* In fact, Roy's solution is not changed by the introduction of a tax rate τ since

$$Pr\{(1 - \tau)\pi \leq 0\} = Pr\{\pi \leq 0\},$$

where, as before, $\pi = px - c - f$. Telser's problem becomes

$$\max (1 - \tau)E(\pi), \text{ subject to } Pr\{\pi \leq 0\} \leq \alpha,$$

which is equivalent to (8), and Kataoka's problem

$$\max z, \text{ subject to } Pr\{(1 - \tau)\pi \leq z\} \leq \alpha,$$

requires maximizing

$$z = (1 - \tau)[(g + hQ)x - TC],$$

which has the same solution as (9).

A tax rate increase will decrease output under Allais' criterion. This follows from the first and second-order conditions since

⁹ See footnote 7.

$$\frac{dx}{d\tau} = \left(\frac{\partial^2 \Phi}{\partial x^2} \right)^{-1} (\overline{MR} - MC) < 0.$$

A *lump sum tax* has the same effect on profits as a fixed cost increase. Therefore, output will increase, decrease or remain constant for firms behaving according to criteria (1), (2) or (3), respectively. It may increase or decrease in the case of (4).

A *specific sales tax* is equivalent to an additive downward shift in the expected demand curve. Therefore, from our previous analysis we conclude that the change in Roy's solution will have the same sign as $h'(x)$, and that the outputs of Telser's, Kataoka's and Allais' firms will decrease. An *ad valorem sales tax* is equivalent to a multiplicative downward shift in the expected demand curve, for which it is straightforward to show that the change in Roy's solution will have the same sign as $gh' - g'h$. In particular, it will not change in the competitive case ($g' = h' = 0$) and under multiplicative uncertainty ($g = g' = 0$), and will increase in the monopolistic-additive uncertainty case ($g' < 0, h' = 0$). Moreover, when subject to an *ad valorem* tax, Kataoka's firm will always reduce output. Telser's and Allais' firms will reduce output when $gh' - g'h \leq 0$ (which includes competition and multiplicative uncertainty), but *might* increase output otherwise.

4. CONCLUSIONS

The first conclusion of this study is that the output of the quantity-setting firm under price uncertainty will be lower than under certainty. This result is interesting since the sensitivity of output to uncertainty reducing information has important welfare implications, but it is not new. The same result has been derived under the expected utility-risk aversion hypothesis. The comparative statics analysis of optimal output under profit-safety preferences yielded more interesting and novel results. In particular, it showed that, in an uncertain world, apparently minor changes in preferences may imply radically different patterns of behavior. The expected response of optimal output to changes in fixed costs, price and taxes is summarized in Table 1 for the profit-safety preferences studied in this paper as well as for the traditional certainty model and the expected

TABLE 1
COMPARATIVE STATICS OF OPTIMAL OUTPUT

Increases in	Response of Optimal Output					
	Certainty	Roy	Telser	Kataoka	Allais	Utility
Fixed costs	0	+	—	0	?	—
Price	+	?	+	+	+	+
Proportional profit tax	0	0	0	0	—	?
Sales taxes	—	?	—	—	—	—

utility-decreasing absolute risk aversion hypothesis.¹⁰

Several conclusions can be derived from this table:

(1) Maximizing the certainty equivalent profit implied by Kataoka's criterion has the same comparative statics properties as the certainty model. That is, not all reasonable preference models which allow for risk aversion¹¹ contradict the predictions of the certainty model.

(2) The predicted sign of the response of output to an increase in fixed costs (produced by a lump sum tax, for instance) depends on the choice of the preference model. All the models but Roy's give the same predictions for price and sales tax changes. The response of output under Roy's criterion and under the expected utility-increasing absolute risk aversion hypothesis to changes in fixed costs, price and sales taxes are similar.¹²

(3) Recent analysis has qualified the well-known proposition of no response to proportional profit taxation. It has been shown that competitive output will change if the utility function does not belong to the constant relative risk aversion class [11]. On the other hand, this paper has shown that safety-first preferences have the same implication as the certainty model. Therefore, we must conclude that the matter cannot be settled on theoretical grounds only.

(4) The comparative statics analysis indicates that an almost complete discrimination among the alternative theories could be achieved with suitable empirical evidence on the firm's past responses to changes in the profit tax rate and changes in lump sum taxes and subsidies.

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¹⁰ See Leland [7] and Sandmo [11]. The response of output to an *ad valorem* sales tax follows from a minor change in their treatment of the increased price effect. Under increasing absolute risk aversion, the output response is reversed for fixed cost changes and may be reversed for price and sales tax changes.

¹¹ Notice that Kataoka's criterion rejects fair assets with negative α -fractile.

¹² See footnote 10.

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