UTILITY ANALYSIS OF CHANCE-CONSTRAINED PORTFOLIO SELECTION

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I. Introduction

Single-period portfolio selection deals with the allocation of an investor's initial wealth to a finite number of risky assets according to his preferences over random final wealth. The purpose of this paper is to study chance-constrained portfolio selection from the point of view of utility theory.

According to utility theory, the investor maximizes the expected utility of final wealth and the portfolio problem is [15], [26]:

(1) $\max \ EU(\Sigma_j \gamma_j X_j)$, subject to $\Sigma_j \gamma_j = 1$, $\gamma_j \geq 0$, $j = 1,\ldots,n$,

where U is the utility function, γ_j is the fraction of the initial wealth invested in asset j, and X_j is the random monetary outcome corresponding to the investment of all the initial wealth in asset j.

The chance-constrained approach assumes that the investor's preferences are representable by the expected value of final wealth and the probability that final wealth will fall below a certain "survival" or safety level s.[1] The chance-constrained problem is [17], [1]:

(2) $\max \ \Sigma_j \gamma_j E(X_j)$, subject to $Pr\left(\Sigma_j \gamma_j X_j < s\right) \leq \alpha$, $\Sigma_j \gamma_j = 1, \gamma_j \geq 0$, $\forall \ j$,

where α is the maximum admissible probability of "ruin." If multiple maxima correspond to a given α, the solution with the lower probability of ruin is selected. Both (1) and (2) assume that random returns per unit of investment

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[1]It is assumed that the investor is forced to change drastically his usual pattern of consumption or operations below the survival level. This is the case of bankruptcy, for instance (see Roy [23, pp. 432-433]).

are independent of the scale of the investment.

The second section of this paper relates the chance-constrained model to utility theory and shows that some of its solutions may be inefficient, that is, nonoptimal for all utility functions in the class implied by its formulation. It also shows how to exclude inefficient solutions. The third section relates the efficiency analysis of chance-constrained choices to the stochastic dominance literature. The fourth section studies the characteristics of chance-constrained solutions when the assets follow a multinormal distribution. The appendix presents some of the properties of the class of utility functions related to the chance-constrained model.[2]

II. Efficient Portfolio Selection According to the Expected Wealth-Probability of Ruin Criterion

A chance-constrained problem with a fixed confidence level α does not admit compensation of a small increase in α by any increase in expected wealth. It is easy to verify that this contradicts the continuity and independence axioms of utility theory [3]. Therefore, as Borch [5, p. 42] has already observed, no utility function can represent the preference ordering of chance-constrained programming with fixed α. The proponents of chance-constrained models admit, however, that the final choice of α may depend on its possible trade-off with expected wealth (see [17]). Therefore, it seems convenient to generate the chance-constrained solutions for all $\alpha \epsilon [0,1]$ and then make the final choice. These solutions will form a pairwise undominated locus in the expected wealth-probability of ruin plane. A portfolio (probability distribution of final wealth W) F will be included in such a locus if and only if there is no portfolio G such that the following inequalities hold and at least one is strict:

(3)
$$E_G(W) \geq E_F(W), \text{ and } G(s) \leq F(s)$$

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[2]The literature on chance-constrained programming has not discussed the choice-theoretic foundation of the model or has wrongly assumed the existence of an unrestricted utility function [18]. Recent analyses of safety-first criteria complement the present paper: Pyle and Turnovsky [20 and 21] have presented a graphical analysis of the relationship between the chance-constrained and expected-utility maximization models in the case of normal assets and Chipman [6] has considered the relationship between safety-first and lexicographic utility. A detailed analysis of safety-first criteria and their implications is made in [3].

where s is the survival point, E is the expectation operator, and G(s) and F(s) are the probabilities that W < s under G and F, respectively.

Having removed the first source of inconsistency, we pass to the study of the pairwise undominated locus from the point of view of utility theory. For that we use the following result due to Markowitz [15, p. 236]:

If an investor orders risky assets solely on the basis of expected wealth E(W) and probability of ruin F(S) and accepts the axioms supporting the expected utility theorem, his preference ordering over the F(W), F(s) combinations is uniquely represented by the positive linear transformations of the function E(W) + cF(s).[3]

This function uniquely implies the following utility function:

(4)
$$U(W) = \begin{cases} W, & \text{for } W \geq s, \\ W+c, & \text{for } W < s, \end{cases}$$

where we restrict the constant c to negative values in order to get an increasing function. It is shown in the appendix that (4) has a number of desirable properties, for instance: it implies risk aversion whenever the initial capital is above s; it results in the maximization of the expected monetary value for negligible risks (consistently with management science practice); and it exhibits decreasing absolute risk aversion in the sense that the amount invested in risky assets increases with wealth. Furthermore, this utility function is consistent with the level of aspiration theory developed by Lewin and other psychologists (see Siegel [25]).

Since the loci of constant expected utility curves in the E(W), F(S) plane are linear, the following efficiency conditions follow for any utility function (4) with given s and unknown c: a portfolio F is undominated if and only if no pair of portfolios G and H exists such that the following inequalities hold and at least one is strict:

(5)
$$\gamma E_G(W) + (1-\gamma) E_H(W) \geq E_F(W), \text{ and } \gamma G(s) + (1-\gamma) H(s) \leq F(s),$$

for some $\gamma \in [0,1]$.

The expected value and the probability of ruin on the left-hand sides of (5) correspond to the mixture (compound littery) $[\gamma G, (1-\gamma)H]$, but this mixture does not necessarily have to be feasible for (5) to apply. These

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[3]Markowitz's proof assumes that the distributions have a finite number of outcomes, but his result is valid for arbitrary distributions with finite means.

conditions exclude those portfolios lying on the nonconvex and downward-sloping parts of the envelope of pairwise undominated portfolios in the $E(W)$, $F(s)$ plane. It is easy to see that any portfolio satisfying conditions (5) will always satisfy conditions (3) and, therefore, the solution to the portfolio problem with utility function (4) will be contained in the solution locus generated by the chance-constrained problems for all $\alpha\epsilon[0,1]$.[4] The converse is not true since there are distributions for which the chance-constrained solution locus includes inefficient portfolios. The following example confirms this point: consider two risky assets X_1 and X_2 with joint probability function $p(x_1,x_2)$ as given in Table 1, and let the survival point s be $\frac{1}{2}$. The chance-constrained problems

$$\max_{0\leq\gamma\leq 1} \quad \gamma E(X_1)+(1-\gamma)E(X_2)$$

$$\text{subject to } Pr\left(\gamma X_1+(1-\gamma)X_2<\frac{1}{2}\right)\leq\alpha,$$

$$\text{for all } \alpha\epsilon[0,1],$$

have the solutions given in Table 2. It can be easily verified that the portfolio $\gamma = .67$ does not satisfy conditions (5). Therefore, the chance-constrained criterion is a necessary but not sufficient condition for optimality according to utility theory. Of course, if the investor wants to order portfolios based solely on expected wealth and probability of ruin and to be

TABLE 1

Values of $p(x_1,x_2)$

		X_2			
		0	.25	1	$p(x_1)$
X_1	0	0	.05	.2	.25
	.25	.2	.1	.2	.5
	4	0	.05	.2	.25
	$p(x_2)$	.2	.2	.6	

[4]This can be proved directly by an argument similar to the proof of Everett's main theorem on generalized Lagrange multipliers (GLM) [8]. Actually, ours is the converse of the GLM problem and our function $EU(W) = E(W) + c\ Pr(W<s)$ is the generalized Lagrangian corresponding to the chance-constrained problem. While Everett was interested in generating the solutions lying on the nonconvexities, we want to exclude them (see below).

TABLE 2

Pairwise Undominated Portfolios

α	γ	F(.5)	E(W)
0≤α< .35	no	feasible	solution
.35≤α< .55	.5	.35	.89
.55≤α< .75	.67	.55	.97
.75≤α<1	1	.75	1.12

consistent with utility theory, he can generate the pairwise undominated port-
folios, that is, those satisfying conditions (3), via chance-constrained pro-
gramming, and then exclude the inefficient portfolios by the use of conditions
(5). This can be done by a simple numerical or geometric search along the
chance-constrained solution locus in the E(W), F(s) plane.

Since the indifference map on the F(W),F(s) plane is a family of paral-
lel straight lines with positive slope $-\frac{1}{c}$, the efficient portfolios are those
which correspond to the points of tangency between the chance-constrained solu-
tion locus and the indifference lines as their slope varies over (o,∞). If
the locus is strictly concave, only its endpoints will be efficient. For a
sufficiently high survival level s no feasible solution will usually exist for
α = 0. Moreover, no efficient solution will exist for α = 1, at least that
max E<s for all α. Finally, any indifference map with abscissas E>s is not
defined at α = 1.

III. Relationship between the Utility Analysis of Chance-Constrained
Portfolio Selection and Stochastic Dominance

Stochastic dominance deals with the ordering of risky assets under
rather general conditions. The basic result was presented some time ago by
Massé and Morlat [16] under the name of *principle of absolute preference*. It
says that a distribution G is preferred or indifferent to a distribution F for
any nondecreasing utility function and strictly preferred for some, if and only
if G(w) ≤ F(w) for all w ε W, and G(w) < F(w) for some w ε W, where W is the
set of possible monetary outcomes. In this case it is said that G stochas-
tically dominates F. This result has been successively rediscovered by Quirk
and Saposnik [22], Hadar and Russell [11] and Hanoch and Levy [12]. The last
two papers also weakened the dominance condition by tightening the requirements
on the utility function. Specifically, they showed that G dominates F on the
class of nondecreasing concave utility functions if G ≠ F and the area under
G is less than or equal to the area under F for all w ε W. Hadar and Russell

have conveniently called these two conditions stochastic dominance of the first degree and stochastic dominance of the second degree, respectively. Obviously, first-degree stochastic dominance holds for all utility functions in the class (4). More interesting is the fact that the pairwise dominance conditions (3) are weaker than first degree but different from second-degree stochastic dominance. Conditions (3) are necessary and sufficient for the dominance of G over F on the class of utilities (4) since

$$(6) \qquad E_G(U) - E_F(U) = \int_{-\infty}^{\infty} w[dG(w) - dF(w)] + c[G(s) - F(s)],$$

or, integrating by parts,

$$(7) \qquad E_G(U) - E_F(U) = \int_{-\infty}^{\infty} [F(w) - G(w)] \, dw + c[G(s) - F(s)].$$

That is, G dominates F if and only if the *total* area under G is less than or equal to (is less than) the *total* area under G, and the probability of ruin under G is less than (or equal to) the probability of ruin under F. First-degree stochastic dominance implies these conditions but not vice versa.

In the previous section we showed that conditions (3) are only necessary and that (5) are necessary and sufficient when more than two assets are simultaneously considered. This improvement followed directly from the simple nature of (4). In general, the smaller the class of utilities under consideration, the weaker the efficiency conditions.

IV. The Case of Normally Distributed Assets

The nice properties of the normal distribution have made it the standard assumption in the practice of chance-constrained programming. It is well known that, under normality, expected utility is always a function of two parameters which, by the implicit function theorem, can be chosen to be $E(W)$ and $F(s)$. Then, it follows that the resulting indifference curves in the (E, F) plane will not be linear for utility functions other than (4). This does not contradict Markowitz's result because, in general, for nonnormal assets and preference orderings other than those represented by (4), the investor will consider other characteristics of the distribution besides or instead of $E(W)$ and $F(s)$. That normal assets can be ordered by nonlinear indifference curves in the (E, F) plane is not surprising. Markowitz's theorem is based upon the requirements that utility theory imposes on mixtures (the continuity and independence axioms) and, under the normality assumption, mixtures are not possible since they would result in nonnormal assets.

Strictly, utility theory and our previous results would be irrelevant in a world where only normal assets are possible, but the consistency of ranking criteria with utility theory is a very relevant question when normality is only a convenient assumption for a subset of decisions, as is the case in the practice of chance-constrained portfolio selection. Thus, we now show that *the solution locus generated by the chance-constrained problem (2) for all $\alpha \epsilon [0,1]$ is efficient in the sense of conditions (5) if the assets follow a multinormal distribution with means above the survival level s.*[5]

A direct proof of this result is easy but tedious. Fortunately, it can also be established using the familiar (μ,σ) plane. It is well known that the (μ,σ) opportunity locus is always convex. Moreover, there is a one-to-one correspondence between this locus and the chance-constrained solutions under the normality assumption (see [20]). This locus will be efficient in the sense of conditions (5) if the indifference curves corresponding to utility function (4) are strictly concave in the (μ,σ) plane. This, in fact, can be shown to be so.[6] Therefore, we conclude that the (μ,σ) criterion is a necessary and sufficient condition for efficiency when the assets are normally distributed and the utility function is linear with a jump discontinuity. This complements the well-known result on (μ,σ) efficiency for normal assets and concave utility functions [12].

It should be noted that, under normality, the chance-constrained approach does not provide a truly different alternative to the (μ,σ) efficiency

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[5]The last requirement still admits the most relevant cases. Actually, it is sufficient to require that at least one mean be greater than s. If all the means are below s, the solution set may be concave, but that is not a very interesting case.

[6]The equation of an indifference curve on the (μ,σ) plane is $\mu - cF(z) = u$, where $z = (s - \mu)/\sigma$ and u is a constant. Let (μ_1,σ_1) and (μ_2,σ_2) be on the same indifference curve and $\mu_2 > \mu_1 > s$. The curve will be strictly concave if any convex combination

$$\mu_\gamma = \gamma\mu_1 + (1-\gamma)\mu_2, \quad \sigma_\gamma = \gamma\sigma_1 + (1-\gamma)\sigma_2, \quad 0 < \gamma < 1,$$

belongs to a higher indifference curve. This will be so if

$$F(z_\gamma) < \gamma F(z_1) + (1-\gamma) F(z_2),$$

where $z_\gamma = (s-\mu_\gamma)/\sigma_\gamma$, which is true since F(z) is strictly convex over $(-\infty, 0]$ and $z_1 < z_\gamma < z_2$.

analysis, because of the one-to-one correspondence between the chance-constrained solutions and the (μ,σ) opportunity locus.

V. Conclusion

We have shown that, if an investor accepts the expected utility theorem, he should not use chance-constrained programming with a fixed α. Firstly, there is no specific α level which can be said *a priori* to be optimal for a particular investor. That level will result from the simultaneous consideration of the locus of efficient solutions and the utility function. Secondly, the solution(s) obtained for a specific α (or for a certain set of α values) may be inefficient in the sense of being nonoptimal for the class of utility functions implied by the expected wealth-probability of ruin criterion. Therefore, it is necessary to derive the complete chance-constrained solution locus in order to obtain efficient portfolios of its convexity cannot be established *a priori*. Finally, we have shown that the chance-constrained solutions are always efficient when the assets follow a multinormal distribution but, in this case, it is well known that the opportunity locus derived via chance-constrained programming is the same as the (μ,σ) opportunity locus.

APPENDIX

Some Properties of Linear Utility
Functions with a Jump Discontinuity

Choices based solely on expected wealth and the probability of ruin but subject to the consistency requirements of utility theory imply a linear utility function with a jump discontinuity. We have seen that this type of function, henceforth called Λ-function, permits eliminating those chance-constrained solutions that are inefficient according to utility theory and, at the same time, provide a choice-theoretic justification to the chance-constrained approach.

In this appendix we show that Λ- functions are plausible representation of preferences according to the theory of risk aversion. We do this by analysis of the behavioral implications of this type of utility functions rather than by the use of direct empirical evidence.[7] This indirect approach was started by Bernoulli [4] and is the one followed in the most relevant writings on the subject, including those of Markowitz [15], Roy [23], Pratt [19] and Arrow [2]. It is also by this method that doubts have been cast on the plausibility of quadratic utility (see [2] and [19]).

An individual is said to have global (local) risk aversion if he, being in a (some) certain position, would not buy an actuarially fair asset. It is a direct consequence of Jensen's inequality that global risk aversion exists if and only if the utility function is strictly concave. It follows also that there exists local risk aversion in those domains over which the utility function is strictly concave. The converse is not true, however, since:

Property 1. *An individual with a Λ-utility function and initial wealth W_o at or above the survival point s will never buy an actuarially fair asset with positive probability of ruin.*

This is obvious, since for $E(W) = W_o$ and $Pr(W < s) > 0$,

(P-1) $$EU(W) = E(W) + c \, Pr(W < s) < W_o.$$

Notice that, if $W_o < s$, the individual will have risk preference. Markowitz [15, p. 295] considered this a limitation of Λ-functions without realizing that risk bearing may be better than no change for those aspiring to achieve a minimum subsistence level or a drastically higher wealth level. Research on the

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[7]Some empirical evidence in favor of this type of utility function has been provided by the tests of the Lewinian level of aspiration theory (see Siegel [25]).

economic behavior of Pakistani farmers [13] and American oil wildcatters [10]
supports this point. At any rate, the $W_o <$ s case does not apply to the typi-
cal investor.

Property 2. *An individual with a Λ-utility function will always buy an
actuarially favorable asset.*

This is the counterpart of a similar result for concave utility func-
tions (see Arrow [2, p. 99]). In order to establish it, we have to consider
the $W_o \geq$ s case only (otherwise the individual will have risk preference and,
a fortiori, will always buy a favorable asset). Suppose that the individual
can invest the amount k, $0 \leq k \leq W_0$, in an asset with random return R independent
of k. Let us assume that the distribution of R has finite mean E(R), and
either has finite variance V(R) or is stable with infinite variance.[8] The
final capital will be $W = W_o + kR$. If R is bounded from below, the probability
of ruin will be zero for a sufficiently small but positive k, but for the
general case it is required that

$$EU(W) = W_o + k\ E(R) + c\ Pr\left(W_o + kR < s\right) > W_o,$$

for some k > 0. This will be so if

(A1) $$\frac{1}{k} Pr\left(R < \frac{s - W_o}{k}\right) \to 0 \text{ as } k \to 0.$$

When V(R) exists, Tchebycheff inequality gives the following upper bound of
(A1)

$$\frac{kV(R)}{[s - W_o - kE(R)]^2} \to 0 \text{ as } k \to 0.$$

Moreover, for all stable distributions with finite mean and infinite variance,
(A1) has the following upper bound (see Mandelbrot [14, pp. 398-399]):

$$\frac{ck^{\alpha-1}}{(W_o - s)^{\alpha}} \to 0 \text{ as } k \to 0,$$

where c > 0 and $\alpha\varepsilon$ (1,2) are constants.

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[8]See Fama [9] for a discussion of the relevance of stable distributions
with infinite variance.

Property 3. Λ-*functions exhibit decreasing absolute risk aversion.*

Arrow [2] and Pratt [19] have shown that decreasing absolute risk aversion is a desirable property. For differentiable functions absolute risk aversion is measured by the ratio $r_A = - U''(W)/U'(W)$. According to the Arrow-Pratt analysis, if r_A is increasing, the amount invested in a favorable risky asset will decrease with wealth (it is well known, for instance, that quadratic utility exhibits this objectionable behavior).

Λ-functions are not differentiable. Nevertheless, they imply decreasing absolute risk aversion in the sense that the amount invested in a favorable asset increases with wealth. In fact, we now show that $\frac{dk}{dW_o} > 0$ for any twice differentiable distribution $F(R)$. From Property 2 we know that $k > 0$. Consider the nontrivial case where R is not bounded from below and $k < W_o$: the conditions for a maximum of $EU(W)$ are

(A2) $E(R) + (-cz/k)\ f(z) = 0$

(A3) $-z\ f'(z)\ -2\ f(z) > 0,$

where $z = (s - W_o)/k$. Differentiating (A2) with respect to W_o and solving for $\frac{dk}{dW_o}$ we get

$$\frac{dk}{dW_o} = [f'(z) + \frac{1}{z} f(z))]/[-zf'(z) - 2f(z)] > 0,$$

by condition (A3).

Samuelson [24] has shown that, if the utility function is strictly concave and the assets are identical and independently distributed with finite variance, maximum expected utility is obtained by an even allocation of the initial wealth among the assets. He has also shown that diversification pays if the assets have the same mean and at least one is independently distributed from the rest. Properties 4 and 5 show that diversification pays for Λ-utility functions.

Property 4. *If the assets follow independent stable distributions differing only in the scale parameter* $\epsilon_j^\alpha > 0$, *j=1,..,n, with characteristic exponent* $\alpha > 1$ *(that is, with a finite common mean* δ *) and preferences are represented by an* Λ-*function with survival point* $s<\delta$, *expected utility has a*

1003

unique maximum at $\gamma_j = \dfrac{\epsilon_j^{\alpha/(1-\alpha)}}{\Sigma_i \epsilon_i^{\alpha/(1-\alpha)}}$, $\forall\ j$, where γ_j is the fraction of the

initial capital invested in asset j.[9]

Property 4 simply says that the amount invested in a given asset is an increasing function of the "precision" of its distribution (that is, of the inverse of its scale parameter). When $\epsilon_j^\alpha = \epsilon^\alpha, \forall\ j$, $\gamma_j = \dfrac{1}{n}$. Moreover, when

$\alpha = 2$, $\epsilon_j^\alpha = \dfrac{\sigma_i^2}{2}$ and $\gamma_j = \dfrac{\sigma_j^{-2}}{\Sigma_i \sigma_i^{-2}}$, where σ_j^2 is the variance of the normal dis-

tribution.

When the means are equal, the maximum of EU(W) is found by minimizing the probability of ruin which, given the stability assumption, is

(A4) $\Pr(W < s) = F\left[(s - \delta)\ /\ (\Sigma_j \gamma_j^\alpha \epsilon_j^\alpha) \right]^{1/\alpha}$.

The minimum of (A4) is obtained when the scale of the final portfolio $\Sigma_j \gamma_j^\alpha \epsilon_j^\alpha$ is a minimum. Since $\gamma_j \geq 0$ and $\alpha > 1$, the scale is a strictly convex function and

$$\min\ \Sigma_j \gamma_j^\alpha \epsilon_j^\alpha\ ,\ \text{subject to}\ \Sigma_j \gamma_j = 1,\ \gamma_j \geq 0,\ \forall\ j,$$

has a unique solution at $\gamma_j = \dfrac{\epsilon_j^{\alpha/(1-\alpha)}}{\Sigma_i \epsilon_i^{\alpha/(1-\alpha)}}$, $\forall\ j$.

Property 5. *If each asset j has an arbitrary independent distribution F_j with mean $\mu_j = \mu$ and variance $\sigma_j^2 > 0$, and preferences are represented by a Λ-function with survival point $s < \mu$, $\gamma_j = \sigma_j^{-2}/\Sigma_i \sigma_i^{-2}$, $\forall\ j$, is the unique solution to $\max\limits_{\gamma}\ \min\limits_{F_j}\ EU(W)$.*

This is an extension of Roy's [23] result on safety-first diversification. When $\sigma_j^2 = \sigma^2. \forall\ j$, $\gamma_j = \dfrac{1}{n}$. Moreover, if in a set of assets with the same mean and finite variances, one asset, say the first, is independently

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[9]For a discussion of the properties of stable distributions, see Mandelbrot [14].

distributed from the rest, $\gamma_1 = \sigma_1^{-2}/(\sigma_1^{-2} + S^{-2})$ and $0 < \gamma_1 < 1$, where S^2 is the variance of the maximin portfolio of assets $2, \ldots, n$.

Since the means are equal, $\max_{\gamma} \min_{F_j} EU(W)$ is found by

$$\min_{\gamma} \max_{F_j} \Pr\left(W < s\right) = \min \frac{\Sigma_j \, \gamma_j^2 \, \sigma_j^2}{\Sigma_j \gamma_j^2 \sigma_j^2 + (s - \mu)^2}$$

according to Cantelli's inequality (see Cramer [7, p. 256]), or by

$$\min \Sigma_j \, \gamma_j^2 \, \sigma_j^2 \text{ , subject to } \Sigma_j \, \gamma_j = 1, \, \gamma_j \geq 0,$$

which has the unique solution

$$\gamma_j = \sigma_j^{-2}/\Sigma_i \sigma_i^{-2} \text{ , } \forall \, j.$$

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